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TOWARDS AN ALGEBRAIC MULTIGRID METHOD FOR TOMOGRAPHIC IMAGE RECONSTRUCTION – IMPROVING CONVERGENCE OF ART

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Abstract. *In this paper we introduce a multigrid method for sparse, possibly rank-deficient and inconsistent least squares problems arising in the context of tomographic image reconstruction. The key idea is to construct a suitable AMG method using the Kaczmarz algorithm as smoother. We first present some theoretical results about the correction step and then show by our numerical experiments that we are able to reduce the computational time to achieve the same accuracy by using the multigrid method instead of the standard Kaczmarz algorithm.*

1 INTRODUCTION

Tomographic reconstruction is the process of reconstructing an object or its cross section from several images of its projections. In the 2D case the object is illuminated by a fan-beam of X-rays, where the signal is attenuated by the object.

Due to its speed, filtered back projection (FBP)[1,2,3] is still state-of-the-art in 2D and 3D reconstruction for clinical use where time matters. But it is known[4] that an algebraic reconstruction technique needs only one third of X-ray images compared to FBP to reconstruct an image of comparable quality in 3D. Therefore many different algebraic reconstruction techniques like Kaczmarz (ART)[5,6], Cimmino (SART), Censor and Gordon (CAV)[7,8] have been developed.

Within ART the object is represented as a linear combination of basis functions, typically pixels, with some unknown coefficients. This leads to a linear system of equations with a sparse system matrix, because each observation is influenced only by the pixels on the corresponding beam path. If enough data are available, one has an over-determined system, which is solved in the least-squares sense. If, on the other hand, there are not enough data in some region to determine the coefficient values, one is faced with an underdetermined problem. In this case, one solves a regularized version of the problem which supplies the additional constraints.

The drawbacks of all these iterative ART techniques is the computational cost of the iterative formula applied to huge data sets, and that these solvers tend to improve the solution very much only in the first few iterations. In practice the reconstruction of a 256^3 or 512^3 volume and $\Phi = 150$ X-ray images of size 1024^2 is common, and therefore an efficient ART is essential to make it more competitive with FBP.

In this paper, we use these iterative methods as smoothers within a multigrid solver[9,10,11] extending first results for real medical datasets using a straight forward full multigrid technique[12].

AMG algorithms are well defined and analyzed in the classical case – square non-singular systems[13]. However, because of the structure of the system matrix the standard multigrid theory is not applicable in our case. From this view point we now try to adapt the basic steps of an AMG procedure – smoothing and correction – to general least squares problems like (5).

In section 2 we briefly summarize the setup phase in order to construct the projection matrix and the right hand side, section 3 presents some theoretical considerations about the correction step. Our experimental results in section 4 compare the Kaczmarz and the multigrid method for the 2D case and confirm our theoretical results.

2 Tomographic Image Reconstruction

In Figure 1 one can find a schematic setup for tomographic image reconstruction. The object is located in a square region Ω that is discretized by a Cartesian grid of pixels Ω^h covering the whole object that has to be reconstructed. We assume for simplicity that the length of each side of the pixel is h and denote the number of pixels by n. The X-ray attenuation function is assumed to take a constant uniform value x_j throughout the jth pixel, for every pixel $j \in \Omega_j^h = \{1, 2, \ldots, n\}$. We denote the number of rays in one projection by R, the number of projections by P and the number of rays in all projections by $m = RP$. The length of the intersection of the ith ray with the jth pixel is then a_{ij} for all $i \in \Omega_i^h = \{1, 2, \ldots, m\}, j \in \Omega_j^h$. a_{ij} therefore represents the contribution of the jth pixel to the total attenuation along the ith ray and is computed via alpha-clipping. b_i is the total attenuation along the ith ray representing the line integral of the unknown attenuation function along the path of the ray. Thus the discretized model can be written as a system of linear equations

$$Ax = b \ , \quad \sum_{j \in \Omega_j^h} a_{ij} x_j = b_i \quad (i \in \Omega_i^h) \ . \tag{1}$$

We call $b \in \mathbb{R}^m$ the measurement vector, $x \in \mathbb{R}^n$ the image vector and $A \in \mathbb{R}^{m \times n}$ the projection matrix. In what follows we shall denote by $(A)_i, (A)^j, (A)_{ij}, \ A^T, N(A), R(A), A^+$

the i-th row, j-th column, (i, j)-th element, transpose, null space, range and Moore-Penrose pseudo inverse of A, respectively. For a given vector subspace $E \subset I\!R^q$, $P_E(x)$ will be the orthogonal projection onto E of an element $x \in I\!R^q$, and $E^\perp$ will denote its orthogonal complement with respect to the euclidean scalar product and norm, denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$. The following properties are known[14]

$$AA^+A = A,\ A^+AA^+ = A^+,\ (AA^+)^T = AA^+,\ (A^+A)^T = A^+A,$$

$$(A^T)^+ = (A^+)^T, \tag{2}$$

$$P_{R(A)} = AA^+,\ P_{R(A^T)} = A^+A,\ P_{N(A^T)} = I - AA^+,\ P_{N(A)} = I - A^+A, \tag{3}$$

where I are the corresponding unit matrices and

$$N(A^+) = N(A^T),\quad R(A^+) = R(A^T). \tag{4}$$

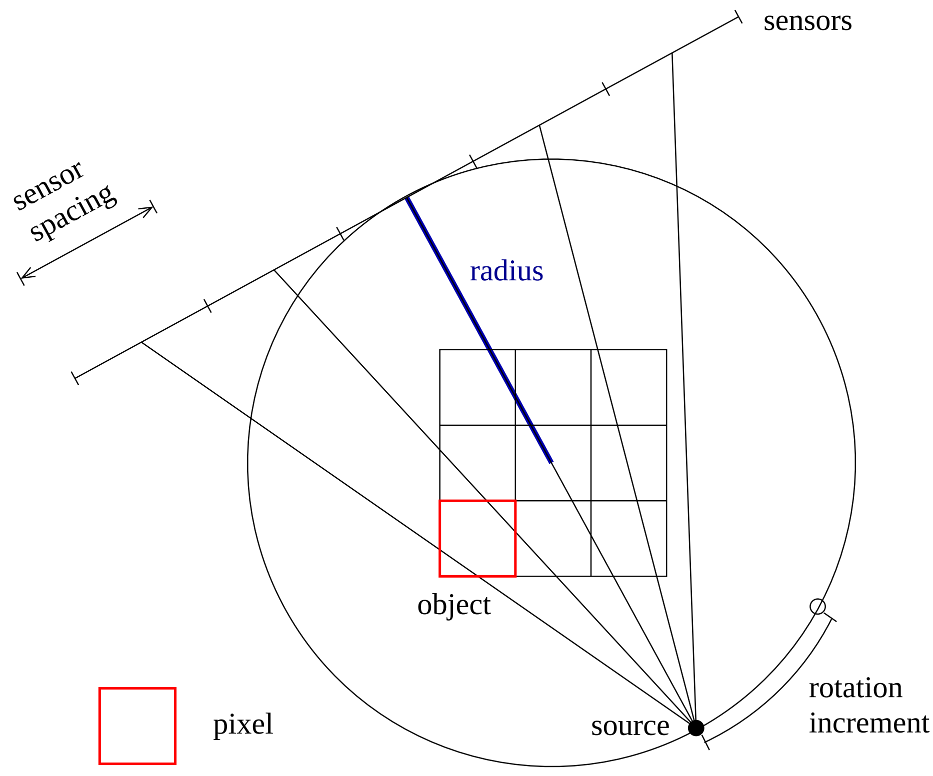

Figure 1: Setup and construction of projection matrix

The matrix A has the property that it is very sparse, since each row of it contains the alpha-clipping values of one ray and a single ray hits only a few pixels. Therefore we have to solve a large, sparse, possibly rank-deficient and inconsistent (e.g. because of measurement errors) least squares(LS)-problem of the form

$$\| Ax - b \| = \min! \tag{5}$$

Let $LSS(A;b), x_{LS}$ be its solutions set and the (unique) minimal norm one, respectively. It is well known that[14]

$$x_{LS} = A^+ b \in N(A)^\perp = R(A^T), \quad LSS(A;b) = x_{LS} + N(A). \tag{6}$$

Usually in the reconstruction problem we are looking for x_{LS} (although this is not always the best possible choice. It can be shown[15] that, because of the rank-deficiency of A, x_{LS} can be sometimes very far from the exact image). In this sense our principal goal in this paper will be to derive a fast and accurate iterative solver for (5).

3 A Multigrid Method for Tomographic Image Reconstruction

3.1 The correction step

We have already analyzed[16] the smoothing properties of Kaczmarz-like algorithms for (5) and have derived results that generalize the well known classical ones by Brandt and Stueben-Ruge[13], we are concerned in the present one with the correction step. To this end we did the following:

1. We propose the following form of the correction step for (5): let $p < n$ be a fixed integer, A_p $(m \times p)$ - the coarse grid matrix and I_p^n $(n \times p)$ - the interpolation operator (which we suppose to be full column rank), then

$$\begin{cases} d = b - Ax \\ \| A_p v_p - d \| = \min! \Rightarrow \quad v_p = A_p^+ d = A_p^+ \left(P_{R(A_p)}(d) \right) \\ \bar{x} = x + I_p^n v_p \end{cases} . \tag{7}$$

2. We introduce new definitions and considerations as follows. For a vector $z \in I\!R^n$ we shall denote by $s(z)$ the solution vector (see (6))

$$s(z) = P_{N(A)}(z) + x_{LS} \in LSS(A;b). \tag{8}$$

Moreover, if

$$b_A = P_{R(A)}(b), \quad b_A^* = P_{N(A^T)}(b), \tag{9}$$

we know that[14]

$$x \in LSS(A;b) \Leftrightarrow Ax = b_A. \tag{10}$$

According to (9), the correction vector v_p in (7) satisfies

$$A_p v_p = P_{R(A_p)}(d). \tag{11}$$

3. We obtained, under additional assumptions general algebraic results on the properties of the correction step (18). We shall briefly describe them below:

Assumption 1. *The matrices A, A_p and I_p^n satisfy the equality*

$$A_p = AI_p^n. \tag{12}$$

Let $x, \bar{x}$ be the approximation before and after the correction step (7), respectively and define the corresponding errors (see (8)) by

$$e = x - s(x), \quad \bar{e} = \bar{x} - s(\bar{x}). \tag{13}$$

Then, the following results have been proved before[17] .

Proposition 1 *If $x \in LSS(A; b)$, then $\bar{x} = x$.*

Proposition 2 *The correction process* (7) *is idempotent.*

Proposition 3 *Let $r, \bar{r}$ be the residuals before and after the correction step* (7)*, i.e. (see* (13)*)*

$$r = Ae = A(x - s(x)) = Ax - b_A, \quad \bar{r} = A\bar{e} = A(\bar{x} - s(\bar{x})) = A\bar{x} - b_A. \tag{14}$$

Then

$$A_p^T \bar{r} = 0, \tag{15}$$

$$\| \bar{r} \| \quad \leq \quad \| r \| . \tag{16}$$

A special case for the errors in (13) is when we refer to the minimal norm solution of (5), i.e.

$$e' = x - x_{LS}, \quad \bar{e}' = \bar{x} - x_{LS}. \tag{17}$$

Proposition 4 *The following equalities hold.*

$$e' = -A^+ d + P_{N(A)}(x) = A^+ r + P_{N(A)}(x),$$

$$\| e' \| = \| A^+ d \|^2 + \| P_{N(A)}(x) \|^2 = \| A^+ r \|^2 + \| P_{N(A)}(x) \|^2, \tag{18}$$

$$\bar{e}' = -A^+ \bar{d} + P_{N(A)}(\bar{x}) = A^+ \bar{r} + P_{N(A)}(\bar{x}),$$

$$\| \bar{e}' \| = \| A^+ \bar{d} \|^2 + \| P_{N(A)}(\bar{x}) \|^2 = \| A^+ \bar{r} \|^2 + \| P_{N(A)}(\bar{x}) \|^2, \tag{19}$$

where $d, \bar{d}, r, \bar{r}$ are the corresponding defects and residuals.

Beside the above described properties of the coarse grid correction step (7) the following one is compulsory for the convergence analysis of a two grid AMG.

$$P_{N(A)}(\bar{x}) = P_{N(A)}(x). \tag{20}$$

It ensures that after the correction step the new approximation $\bar{x}$ generates an error $\bar{e}$ with respect to the same LSS solution. Else, with each application of (7) the solution according to which the error is computed (see (13)) would be changed. In what follows we shall give three sufficient assumptions for that (20) holds.

Assumption 2. *The matrices A, A_p, I_p^n satisfy the following relation*

$$(A^+A)I_p^n = I_p^n(A_p^+A_p). \tag{21}$$

Assumption 3. *The matrices A, A_p, I_p^n satisfy the following relation*

$$A^+A_pA_p^+A = I_p^nA_p^+A. \tag{22}$$

Assumption 4. *The interpolation I_p^n satisfies the following relation*

$$R(I_p^n) = R(A^T). \tag{23}$$

Proposition 5 *Each of the above assumptions ensures the property* (20).

Proposition 6 *(i) If the matrix A has full column rank, then* (21) *and* (22) *are true.*
(ii) If the interpolation operator is of the form

$$I_p^n = A^TE, \tag{24}$$

for some $m \times p$ matrix E, then (23) *is true.*

Corollary 1 *If* (21) *holds, then the errors e and $\bar{e}$ from* (13) *satisfy*

$$\bar{e} = e + I_p^n v_p. \tag{25}$$

3.2 Coarse grid and intergrid transfer operators

We consider only the 2D case here. Suppose that

$$n = 4p \ , \tag{26}$$

and let $P_1, \ldots, P_n$ be the pixels on the "fine grid". The "coarse grid" is obtained by considering the larger pixels formed (each) by 4 adjacent pixels of the fine grid, $P_1^H, \ldots, P_p^H$ (see Figure 2 (A)).

For any $j \in \{1, \ldots, p\}$ we define $S(j)$ as the set of indices of fine grid pixels that form the coarse grid one P_j^H, i.e.

$$S(j) = \{j_1, j_2, j_3, j_4\}, \quad \forall j = 1, \ldots, p, \tag{27}$$

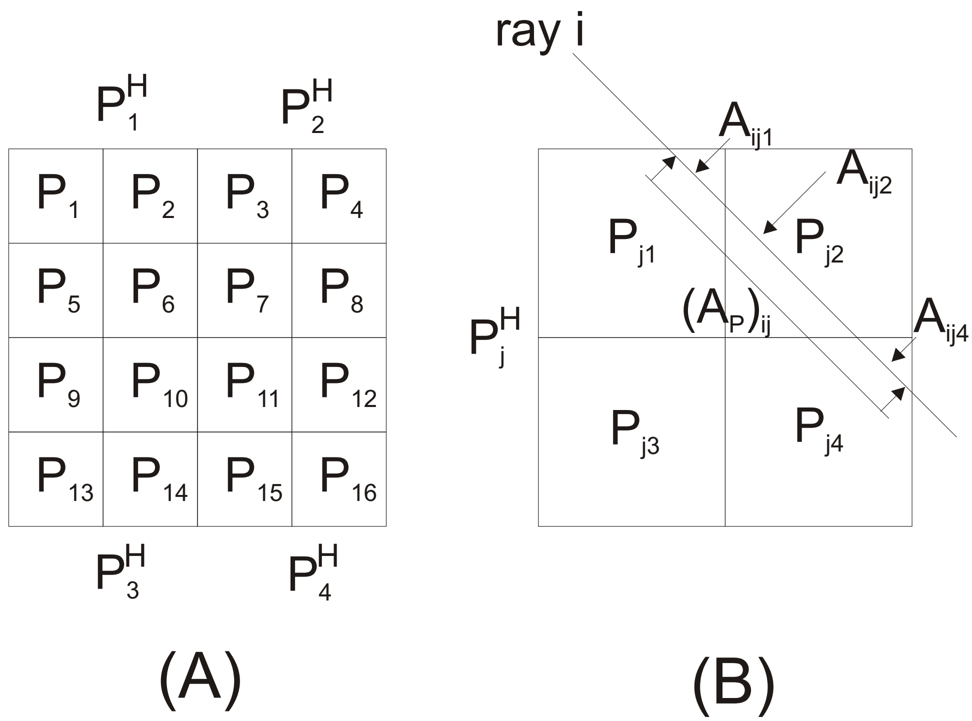

Figure 2: Relation between fine and coarse grid matrix entries.

such that

$$P_j^H = P_{j_1} \cup P_{j_2} \cup P_{j_3} \cup P_{j_4}. \tag{28}$$

We may suppose that the pixels P_i and P_j^H are numbered such that

$$j_1 < j_2 < j_3 < j_4. \tag{29}$$

We construct the above coarse grid matrix A_p following the formulas (see Figure 2 (B))

$$(A_p)_{ij} = \sum_{k \in S(j)} A_{ik}, \ \forall i = 1, \dots, m, \ j = 1, \dots, p \tag{30}$$

and the $n \times p$ interpolation operator I_p^n by

$$(I_p^n)_{ij} = \begin{cases} 1, & \text{if } i \ \in S(j) \\ 0, & \text{if } i \ \notin S(j) \end{cases} \quad , i = 1, \dots, n, \ j = 1, \dots, p. \tag{31}$$

The next two propositions were also proved[17] .

Proposition 7 *The above matrices A, A_p and I_p^n satisfy* (12)*. Moreover, the interpolation operator I_p^n has full column rank.*

Proposition 8 *If A from before has full column rank and A_p and I_p^n are defined as in* (30) *and* (31)*, then A_p has also full column rank.*

Remark 1 *From above it results that our approach* $(30)-(31)$ *satisfies all the properties from section* 2 *when A is full column rank. One disadvantage of this approach would be that the number of rows (i.e. rays) in all the coarse grid matrices remains the same as for A. But, this is still a considerable advantage because, although the dimension m is the same, the number of columns in the coarse grid matrices is divided by* 4 *with each discretization level. On the coarsest level this will result in a matrix for which a problem like* (5) *can be easily solved since it has only very few columns. A very good aspect, beside the above mentioned properties of the correction is that the coarse grid matrices maintain the sparsity of the problem (on the corresponding discretization levels).*

Remark 2 *We considered*[12] *also the possibility to reduce the number of rows (rays) in A_p. This is mathematically equivalent with constructing the coarse grid matrix A_p by (see for comparison* (12)*)*

$$A_p = I_m^q A I_p^n, \tag{32}$$

where I_m^q is an $q \times m$ matrix. It can be a "pick - up" one (i.e. it contains only one 1 *in each row, in a prescribed position) or an "interpolation-like" matrix (as e.g. the linear restriction operator on an* 1D *multigrid for Poisson equation*[13]*) . Unfortunately, such a construction doesn't always satisfy Proposition* 1*, which can destroy the efficiency of the correction process* (7) *(more clearly, the correction step* (7) *is no more compatible with the solutions set $LSS(A; b)$). Indeed, in this case,* (7) *becomes*

$$\left\{ \begin{array}{l} d = b - Ax; \ d^q = I_m^q d \\ \parallel A_p v_p - d^q \parallel = \min! \Rightarrow \quad v_p = A_p^+ d^q = A_p^+ \left(P_{R(A_p)}(d^q) \right) \\ \bar{x} = x + I_p^n v_p \end{array} \right. . \tag{33}$$

Then, if $x \in LSS(A; b)$, by also using that $d = b - Ax = b_A^* \in N(A^T)$ we obtain $d^q = I_m^q b_A^*$, which doesn't always belong to the subspace $N(A^T) \subset N(A_p^+)$, i.e. v_p and $I_p^n v_p$ are not 0*, thus $\bar{x} \neq x$.*

Remark 3 *The $m \times p$ matrix E from* (24) *can be a "pick-up" one, i.e. only with 1's and 0's as entries, defined in the following way (see* $(27)-(28)$*)*

$$(E)_{ij} = \left\{ \begin{array}{ll} 1, & \text{if the } i-th \text{ ray intersects at least} \\ & \text{one pixel } P_k \text{ with } k \in S(j) \\ 0, & \text{else.} \end{array} \right. \tag{34}$$

In this way we can get an (enough) sparse interpolation operator I_p^n. But, in order to keep the Assumption 1, we have to define the coarse grid matrix A_p as in (12)*, which gives us (see* (24)*)*

$$A_p = A I_p^n = A A^T E. \tag{35}$$

Then, beside the fact that its elements have to be computed as scalar products of the form $\langle A_i, A_j \rangle$, the sparsity structure can be different that one for A_p from section 5.1 (see (30)).

4 Numerical experiments

We already know from the 3D case[12] that multigrid clearly reduces the computational time for reconstruction. Here we concentrate on supporting our theoretical results. Therefore we have implemented the setup of the projection matrix and the solution of the least squares problem in 2D in Matlab.

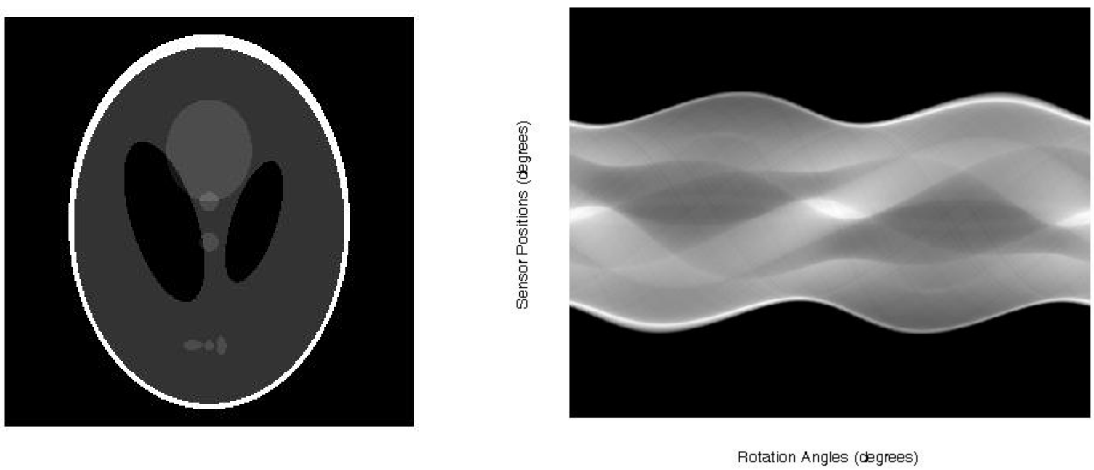

Figure 3: Shepp-Logan phantom (left) and its sinogram (right).

Figure 3 shows the original image (size 256^2), a Shepp-Logan phantom available in Matlab and the corresponding right hand side b, the sinogram, computed by the Matlab routine fanbeam, that uses a setup as shown in Figure 1.

For our experiments we resized the image to $n = 24^2$ and used $m = RP = 1560$ rays in all projections with $R = 39$ and $P = 40$. We set the radius to 40, the sensor spacing to 1 and the rotation increment to 9 (see Figure 1). The structure of the projection matrix $A \in \mathbb{R}^{1560 \times 576}$ that has full column rank for this setup is shown in Figure 4.

The results in Figure 5 show that the multigrid method can reduce both the error (the L_2-norm of the difference between original image and reconstructed image) and the residual (the L_2-norm of $b - Ax$) faster than standard Kaczmarz.

However, we can not expect the usual multigrid convergence rates for elliptic PDEs in this general case. In the medical application it is sufficient to use only a few Kaczmarz sweeps to get acceptable visual results. This can be seen in Figure 6. Due to the huge size of the real problems even only one Kaczmarz sweep on the finest level can take several minutes, and therefore the computational time can be reduced drastically by saving only a few sweeps on the finest level.

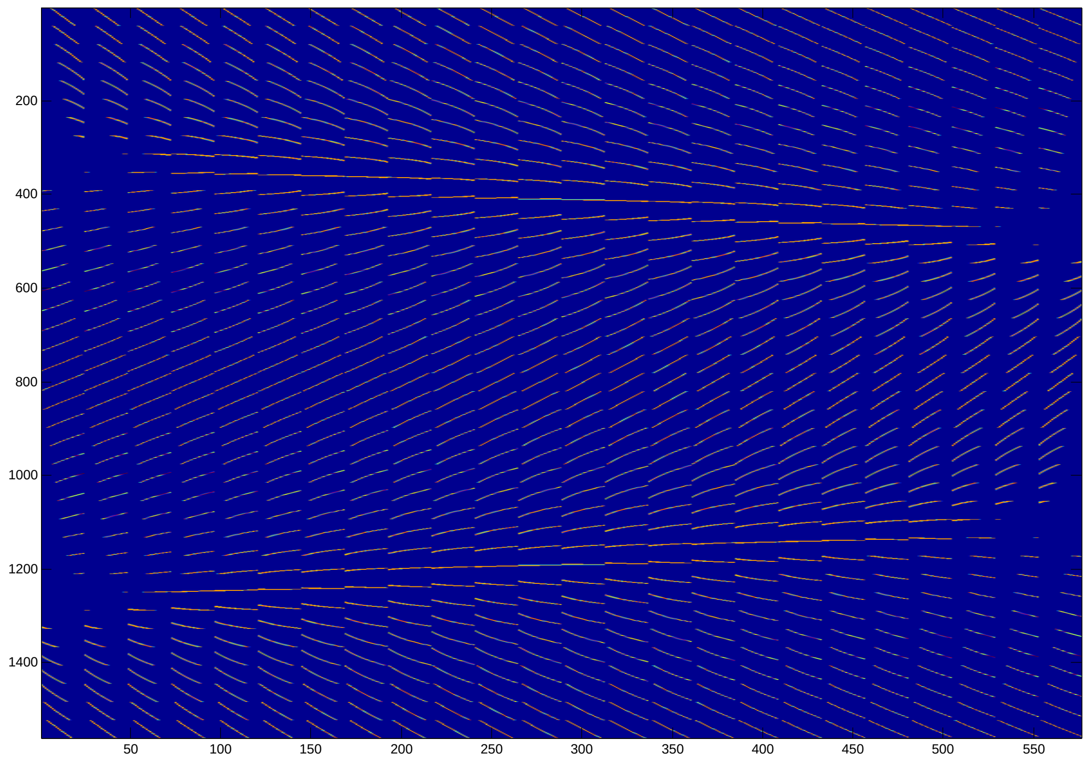

Figure 4: Structure of projection matrix $A \in \mathbb{R}^{1560 \times 576}$. Black corresponds to zero entries.

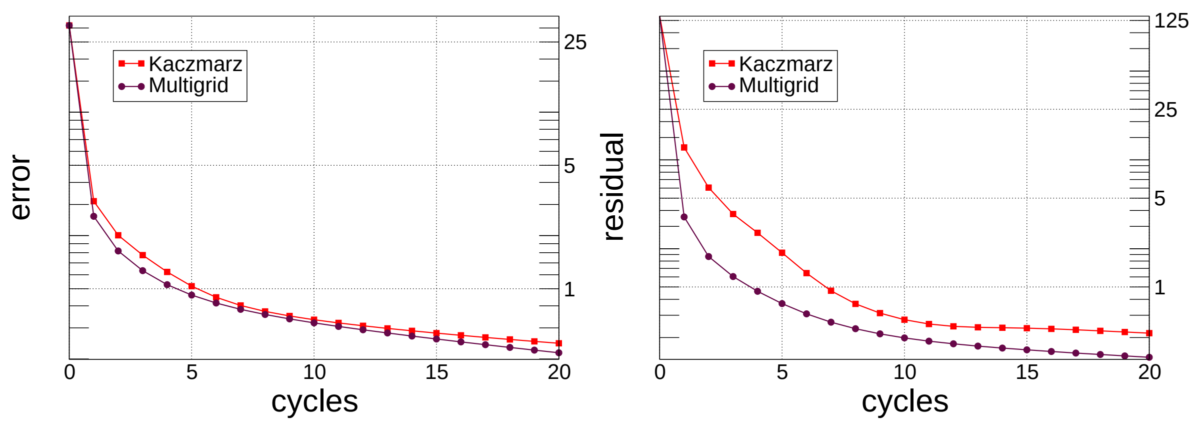

Figure 5: L_2-norms of errors and residuals for V(2,0)-cycles using only 1 level (Kaczmarz) and using 2 levels with a direct solver (pseudo inverse) on the coarse grid.

5 CONCLUSIONS

We have introduced an AMG method for tomographic image reconstruction. The next steps will be the extension of the theory to the general inconsistent case as far as possible

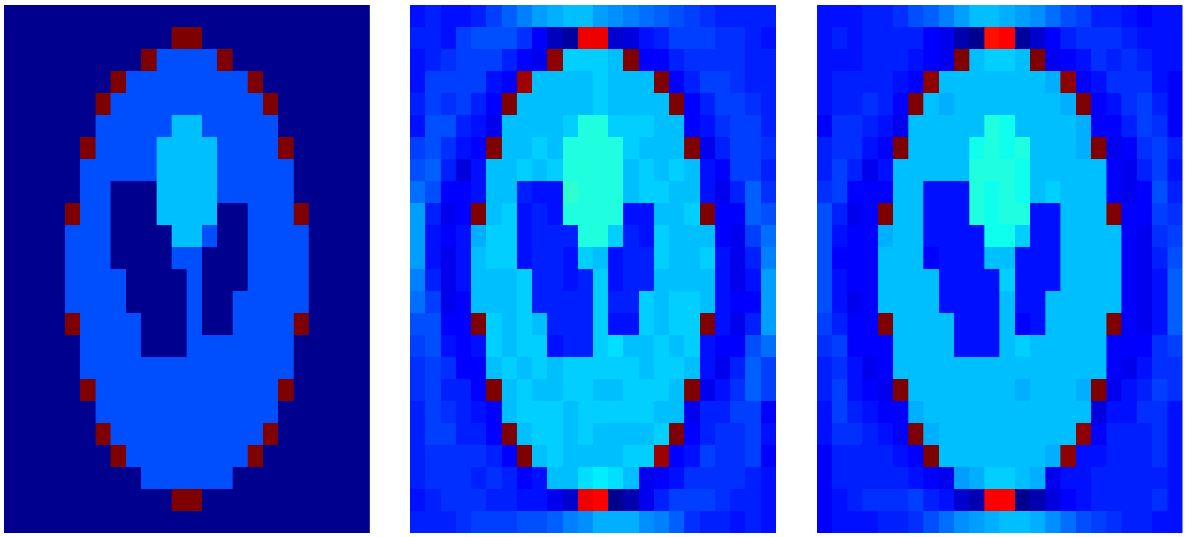

Figure 6: Exact image (left) compared to the reconstructed images using 10 Kaczmarz steps (middle) and 5 V(2,0) cycles (right).

and the deeper evaluation of the method for real medical data sets.

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