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REGULARIZING A TIME-STEPPING METHOD FOR RIGID MULTIBODY DYNAMICS

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Abstract. *Rigid multibody dynamics with friction and impact are ill-posed. The rigidity assumption can be responsible for large null-spaces. By describing the multibody system with springs switching on and off in a variational inequality setting, the solutions are smooth and the behaviour in the stiff limit can be analyzed. From the limiting process it becomes apparent that the regularized solution converges to a unique weighted minimum norm solution of the unregularized problem.*

1 INTRODUCTION

Rigidity is a convenient idealization tremendously reducing the degrees of freedom in a multibody system. However, this idealization involves a limiting process introducing ambiguities when approaching the stiff limit. Whereas a deformable table typically distributes the reaction forces uniformly to its legs, a rigid table has one degree of freedom in how to distribute the reaction forces [1]. For non-dissipative systems the resulting velocity solution is the same. However, if friction is involved neither the forces nor the velocities are usually unique.

A possible approach to regularize the system is to assume that the contact regions are deforming considerably more than the rest of the bodies. This assumption is reasonable e.g. for hard spherical objects as observed in [2]. In the following, these contact deformations will be modeled by contact springs abiding Hooke's Law. The discretization of the system will lead to a nonsingular system. When driving the spring constants homogeneously to infinity the solution converges to a unique solution of the rigid system, which can be characterized as a weighted minimum norm solution of the corresponding rigid problem.

Sec. 2 briefly introduces the differential system and a semi-implicit time discretization for it. Sec. 3 discusses the modeling of bilateral joints, frictionless and frictional unilateral contacts with constraint springs. Sec. 4 summarizes the results and defines open problems.

2 DIFFERENTIAL SYSTEM

The multibody system consisting of n_b bodies can be described by the state variables $\boldsymbol{q}(t) : \mathbb{R} \to \mathbb{R}^{3n_b}$, $\boldsymbol{\varphi}(t) : \mathbb{R} \to \mathrm{SO}(3)^{n_b}$, $\boldsymbol{v}(t) : \mathbb{R} \to \mathbb{R}^{3n_b}$ and $\boldsymbol{\omega}(t) : \mathbb{R} \to \mathbb{R}^{3n_b}$. $\boldsymbol{q}$ denotes the global position of the bodies' centers of gravity, $\boldsymbol{\varphi}$ describes the orientations, $\boldsymbol{v}$ the linear velocities, and $\boldsymbol{\omega}$ the angular velocities. The orientation e.g. can be described by quaternions. The equations of motion lead to the system of ordinary differential equations

$$\begin{aligned} \begin{pmatrix} \dot{\boldsymbol{q}}(t) \\ \dot{\boldsymbol{\varphi}}(t) \end{pmatrix} &= \begin{pmatrix} \boldsymbol{v}(t) \\ \mathbf{Q}(\boldsymbol{\varphi}(t))\boldsymbol{\omega}(t) \end{pmatrix} \\ \begin{bmatrix} \hat{\mathbf{M}} & \\ & \hat{\mathbf{I}}(t) \end{bmatrix} \begin{pmatrix} \dot{\boldsymbol{v}}(t) \\ \dot{\boldsymbol{\omega}}(t) \end{pmatrix} = \mathbf{M}(t) \begin{pmatrix} \dot{\boldsymbol{v}}(t) \\ \dot{\boldsymbol{\omega}}(t) \end{pmatrix} &= \begin{pmatrix} \boldsymbol{f}(t) \\ \boldsymbol{\tau}(t) - \boldsymbol{\omega}(t) \times \hat{\mathbf{I}}(t)\boldsymbol{\omega}(t) \end{pmatrix} + \mathbf{J}(\boldsymbol{q}(t), \boldsymbol{\varphi}(t))^T \boldsymbol{\lambda}, \end{aligned} \tag{1}$$

where $\mathbf{Q}$ is a linear operator depending on the orientations, $\hat{\mathbf{M}} \in \mathbb{R}^{3n_b \times 3n_b}$ is a constant diagonal matrix containing all bodies' masses three times, $\hat{\mathbf{I}} : \mathbb{R} \to \mathbb{R}^{3n_b \times 3n_b}$ is a block-diagonal matrix containing all bodies' inertia tensors, $\mathbf{M} : \mathbb{R} \to \mathbb{R}^{6n_b \times 6n_b}$ is the mass matrix, $\boldsymbol{f} : \mathbb{R} \to \mathbb{R}^{3n_b}$ are external forces, $\boldsymbol{\tau} : \mathbb{R} \to \mathbb{R}^{3n_b}$ are external torques, $\boldsymbol{a} \times \boldsymbol{b}$ is the collection of all cross products between all corresponding three-dimensional vectors in $\boldsymbol{a}$ and $\boldsymbol{b}$, $\mathbf{J} : \mathbb{R}^{3n_b} \times \mathrm{SO}(3)^{n_b} \to \mathbb{R}^{n_c \times 6n_b}$ the constraint Jacobian of n_c constraints and $\boldsymbol{\lambda} \in \mathbb{R}^{n_c}$ is the vector of constraint forces. The differential equations in Eq. (1) can be conveniently discretized by a semi-implicit Euler method. Positions and orientations are integrated implicitly whereas velocities are integrated explicitly leading to a symplectic integration scheme [3]. In the following time-discrete variables are identified by a tilde ˜. Quantities associated with the state of the system after a time step δt are annotated by a tick mark $'$:

$$\begin{pmatrix} \tilde{\boldsymbol{q}}' \\ \tilde{\boldsymbol{\varphi}}' \end{pmatrix} = \delta t \begin{pmatrix} \tilde{\boldsymbol{v}}' \\ \mathbf{Q}(\tilde{\boldsymbol{\varphi}})\tilde{\boldsymbol{\omega}}' \end{pmatrix} + \begin{pmatrix} \tilde{\boldsymbol{q}} \\ \tilde{\boldsymbol{\varphi}} \end{pmatrix} \tag{2a}$$

$$\begin{pmatrix} \tilde{\boldsymbol{v}}' \\ \tilde{\boldsymbol{\omega}}' \end{pmatrix} = \delta t \tilde{\mathbf{M}}^{-1} \begin{pmatrix} \tilde{\boldsymbol{f}} \\ \tilde{\boldsymbol{\tau}} - \tilde{\boldsymbol{\omega}} \times \tilde{\tilde{\mathbf{I}}}\tilde{\boldsymbol{\omega}} \end{pmatrix} + \delta t \tilde{\mathbf{M}}^{-1}\mathbf{J}^T(\tilde{\boldsymbol{q}}, \tilde{\boldsymbol{\varphi}})\tilde{\boldsymbol{\lambda}} + \begin{pmatrix} \tilde{\boldsymbol{v}} \\ \tilde{\boldsymbol{\omega}} \end{pmatrix}. \tag{2b}$$

Note that even though the paper deals with impacts and friction all phenomena are modeled using springs and dampers and thus the positions, orientations and velocities are differentiable. In particular there are no non-smooth position evolutions and velocity jumps on impact. Only in the stiff limit the description in terms of measure differential equations or inclusions becomes mandatory so that the non-smoothness and discontinuities of the solution can be captured [4].

3 MODELING

3.1 BILATERAL JOINTS

A simple bilateral joint can be described by a gap function $\boldsymbol{g_i} : \mathbb{R}^3 \times \mathrm{SO}(3) \to \mathbb{R}^3$ for any joint i. It describes the distance between two points each associated with a body. This gap is supposed to vanish in the stiff limit so that $\boldsymbol{g_i}(\boldsymbol{q}(t), \boldsymbol{\varphi}(t)) = \mathbf{0}$ at all times. In the smooth formulation this hard constraint is replaced by three constraint springs, one for each space dimension, such that

$$\boldsymbol{\lambda_i} = -\operatorname{diag}(\boldsymbol{k_i})\boldsymbol{g_i}(\boldsymbol{q}(t), \boldsymbol{\varphi}(t)), \tag{3}$$

where $\boldsymbol{k_i} \in \mathbb{R}^3$ contains the spring constants and $\lambda_i \in \mathbb{R}^3$ are the constraint springs' restoring forces. Discretizing Eq. (3) implicitly leads to

$$\tilde{\boldsymbol{\lambda}}_{\boldsymbol{i}} = -\operatorname{diag}(\boldsymbol{k_i})\boldsymbol{g_i}(\tilde{\boldsymbol{q}}', \tilde{\boldsymbol{\varphi}}'). \tag{4}$$

Linearization of the gap function $\boldsymbol{g_i}$ around the time at the beginning of the time step results in

$$\begin{aligned} \boldsymbol{g_i}(\tilde{\boldsymbol{q}}', \tilde{\boldsymbol{\varphi}}') &= \boldsymbol{g_i}(\tilde{\boldsymbol{q}}, \tilde{\boldsymbol{\varphi}}) + \delta t \dot{\boldsymbol{g}}_{\boldsymbol{i}}(\tilde{\boldsymbol{q}}, \tilde{\boldsymbol{\varphi}}) \\ &= \boldsymbol{g_i}(\tilde{\boldsymbol{q}}, \tilde{\boldsymbol{\varphi}}) + \delta t \mathbf{J}_{g_i}(\tilde{\boldsymbol{q}}, \tilde{\boldsymbol{\varphi}}) \begin{bmatrix} \mathbf{I} & \\ & \mathbf{Q}(\tilde{\boldsymbol{\varphi}}) \end{bmatrix} \begin{pmatrix} \tilde{\boldsymbol{v}}' \\ \tilde{\boldsymbol{\omega}}' \end{pmatrix} \\ &= \boldsymbol{g_i}(\tilde{\boldsymbol{q}}, \tilde{\boldsymbol{\varphi}}) + \delta t \mathbf{J}_{\boldsymbol{i}*}(\tilde{\boldsymbol{q}}, \tilde{\boldsymbol{\varphi}}) \begin{pmatrix} \tilde{\boldsymbol{v}}' \\ \tilde{\boldsymbol{\omega}}' \end{pmatrix}. \end{aligned} \tag{5}$$

Inserting Eq. (2b) and Eq. (5) into Eq. (4) yields, for a system with joints only, the following linear system of equations (LSE) as time step problems:

$$\frac{1}{\delta t}\boldsymbol{g}(\tilde{\boldsymbol{q}},\tilde{\boldsymbol{\varphi}}) + \left(\mathbf{J}(\tilde{\boldsymbol{q}},\tilde{\boldsymbol{\varphi}})\tilde{\mathbf{M}}^{-1}\mathbf{J}(\tilde{\boldsymbol{q}},\tilde{\boldsymbol{\varphi}})^T + \frac{1}{\delta t^2}\operatorname{diag}(\boldsymbol{k})^{-1}\right)\delta t\tilde{\boldsymbol{\lambda}}$$
$$+\mathbf{J}(\tilde{\boldsymbol{q}},\tilde{\boldsymbol{\varphi}})\left(\begin{pmatrix}\tilde{\boldsymbol{v}}\\ \tilde{\boldsymbol{\omega}}\end{pmatrix} + \delta t\tilde{\mathbf{M}}^{-1}\begin{pmatrix}\tilde{\boldsymbol{f}}\\ \tilde{\boldsymbol{\tau}} - \tilde{\boldsymbol{\omega}} \times \tilde{\hat{\mathbf{I}}}\tilde{\boldsymbol{\omega}}\end{pmatrix}\right) = \mathbf{0} \tag{6}$$

Neglecting the current gap $\boldsymbol{g}(\tilde{\boldsymbol{q}},\tilde{\boldsymbol{\varphi}})$ this LSE is of the type

$$(\mathbf{A}^T\mathbf{A} + \mathbf{D})\boldsymbol{x} - \mathbf{A}^T\boldsymbol{b} = \mathbf{0}, \tag{7}$$

where $\mathbf{A} \in \mathbb{R}^{6n_b \times n_c}$ is possibly rank-deficient, $\mathbf{D} \in \mathbb{R}^{n_c \times n_c}$ is positive and diagonal, $\boldsymbol{x} \in \mathbb{R}^{n_c}$ is the vector of unknowns and $\boldsymbol{b} \in \mathbb{R}^{6n_b}$ is a constant vector. Eq. (7) has a unique solution since $\mathbf{A}^T\mathbf{A}$, which is positive semi-definite (PSD), becomes positive-definite (PD) because of adding the regularization term $\mathbf{D}$. In order to analyze the behaviour of the solution when uniformly increasing the stiffness of the springs a regularization parameter s is introduced scaling the regularization matrix $\mathbf{D}$. The system then approaches the stiff limit for $s \to 0$ in

$$(\mathbf{A}^T\mathbf{A} + s\mathbf{D})\boldsymbol{x} - \mathbf{A}^T\boldsymbol{b} = \mathbf{0}. \tag{8}$$

Such an LSE corresponds to a damped least squares problem and approaches for $s \to 0$ the (unique) *weighted minimum norm* solution $\boldsymbol{x}^*$ of the unregularized system as was proved in detail in [5]:

$$\|\mathbf{D}^{\frac{1}{2}}\boldsymbol{x}^*\|_2^2 = \min_{\mathbf{A}^T\mathbf{A}\boldsymbol{x}=\mathbf{A}^T\boldsymbol{b}} \boldsymbol{x}^T\mathbf{D}\boldsymbol{x} \tag{9}$$

Aside from that the limiting process also involves that the system is ideally damped in the stiff limit. Since the spring is discretized implicitly the time integration introduces numerical damping. Though numerical damping is small for moderate spring constants the damping increases as $s \to 0$.

3.2 UNILATERAL CONTACTS

The gap function can also be used to describe unilateral contacts. But for contacts the gaps in normal and tangential directions are of interest. Subscripts n, t and o denote the normal, tangential and orthogonal components, respectively. A negative normal component indicates penetration of the contacting bodies. Instead of solving a Signorini contact condition in normal direction a constraint spring becomes active as soon as the penetration occurs:

$$g_{i_n}(\boldsymbol{q}(t),\boldsymbol{\varphi}(t)) = -k_i^{-1}\lambda_{i_n}, \qquad \lambda_{i_n} \geq 0 \tag{10a}$$
$$g_{i_n}(\boldsymbol{q}(t),\boldsymbol{\varphi}(t)) \geq -k_i^{-1}\lambda_{i_n} = 0, \qquad \lambda_{i_n} = 0 \tag{10b}$$

Both conditions can be combined in a complementarity problem

$$g_{i_n}(\boldsymbol{q}(t),\boldsymbol{\varphi}(t)) + k_i^{-1}\lambda_{i_n} \geq 0 \perp \lambda_{i_n} \geq 0 \tag{11}$$

or equivalently in a variational inequality problem (VI)

$$\lambda_{i_n} \in [0;\infty), \quad \langle g_{i_n}(\boldsymbol{q}(t),\boldsymbol{\varphi}(t)) + k_i^{-1}\lambda_{i_n}, y_{i_n} - \lambda_{i_n}\rangle \geq 0, \quad \forall y_{i_n} \in [0;\infty). \tag{12}$$

The time-continuous solution would reproduce the behaviour of an elastic collision and no energy would be dissipated. However, discretizing Eq. (11) or Eq. (12) implicitly as before introduces numerical damping again. Thus in the stiff limit the behaviour of an inelastic collision is reproduced instead – even though no damper is present which could dissipate energy.

When modeling (regularized) dry friction in smooth systems the Haff and Werner model [6] is often used. There the friction can for example be modelled as a damper opposing the tangential slip velocity whose damping force is limited in magnitude by $\mu_i(\boldsymbol{z}(t))\lambda_{i_n}$, where $\boldsymbol{z}(t)$ is a shorthand notation for all state variables $\boldsymbol{q}(t)$, $\boldsymbol{\varphi}(t)$, $\boldsymbol{v}(t)$ and $\boldsymbol{\omega}(t)$. Thus a closed frictional contact can be in two states. The static state corresponds to a damping force not at its limits and the dynamic state corresponds to a damping force with magnitude $\mu_i(\boldsymbol{z}(t))\lambda_{i_n}$:

$$\boldsymbol{\lambda}_{\boldsymbol{i}_{t,o}} = -\gamma_i \dot{\boldsymbol{g}}_{\boldsymbol{i}_{t,o}}(\boldsymbol{q}(t),\boldsymbol{\varphi}(t)), \qquad \|\boldsymbol{\lambda}_{\boldsymbol{i}_{t,o}}\|_2 \leq \mu_i(\boldsymbol{z}(t))\lambda_{i_n} \tag{13a}$$

$$\boldsymbol{\lambda}_{\boldsymbol{i}_{t,o}} = -\frac{\dot{\boldsymbol{g}}_{\boldsymbol{i}_{t,o}}(\boldsymbol{q}(t),\boldsymbol{\varphi}(t))}{\|\dot{\boldsymbol{g}}_{\boldsymbol{i}_{t,o}}(\boldsymbol{q}(t),\boldsymbol{\varphi}(t))\|_2}\mu_i(\boldsymbol{z}(t))\lambda_{i_n}, \qquad \|\boldsymbol{\lambda}_{\boldsymbol{i}_{t,o}}\|_2 = \mu_i(\boldsymbol{z}(t))\lambda_{i_n}. \tag{13b}$$

Note that we assume isotropic friction where the damping is the same in both tangential directions and introduce the non-negative damping coefficients $\boldsymbol{\gamma} \in \mathbb{R}^{n_c}$. Assuming λ_{i_n} is known a priori to be $\bar{\lambda}_{i_n}$ then both conditions can be combined in the VI

$$\begin{aligned}\boldsymbol{\lambda}_{\boldsymbol{i}_{t,o}} \in \mathcal{S}(\mu_i(\boldsymbol{z}(t))\bar{\lambda}_{i_n}), \quad \langle \dot{\boldsymbol{g}}_{\boldsymbol{i}_{t,o}}(\boldsymbol{q}(t),\boldsymbol{\varphi}(t)) + \gamma_i^{-1}\boldsymbol{\lambda}_{\boldsymbol{i}_{t,o}}, \boldsymbol{y}_{\boldsymbol{i}_{t,o}} - \boldsymbol{\lambda}_{\boldsymbol{i}_{t,o}}\rangle \geq 0,& \\ \forall \boldsymbol{y}_{\boldsymbol{i}_{t,o}} \in \mathcal{S}(\mu_i(\boldsymbol{z}(t))\bar{\lambda}_{i_n}),&\end{aligned} \tag{14}$$

where $\mathcal{S}(r)$ is the set of vectors in $\mathbb{R}^2$ within a disc of radius r around the origin. This can be seen by checking the two states of the VI. If $\boldsymbol{\lambda}_{\boldsymbol{i}_{t,o}}$ is at the boundary of $\mathcal{S}(\mu_i(\boldsymbol{z}(t))\bar{\lambda}_{i_n})$ then $\|\boldsymbol{\lambda}_{\boldsymbol{i}_{t,o}}\|_2 = \mu_i(\boldsymbol{z}(t))\lambda_{i_n}$ and we expect dynamic friction. The VI then requires that $\dot{\boldsymbol{g}}_{\boldsymbol{i}_{t,o}}(\boldsymbol{q}(t),\boldsymbol{\varphi}(t)) + \gamma^{-1}\boldsymbol{\lambda}_{\boldsymbol{i}_{t,o}}$ is a non-positive multiple of $\boldsymbol{\lambda}_{\boldsymbol{i}_{t,o}}$ and it follows that for $t \leq 0$

$$\begin{aligned}
& \dot{\boldsymbol{g}}_{\boldsymbol{i}_{t,o}}(\boldsymbol{q}(t),\boldsymbol{\varphi}(t)) + \gamma_i^{-1}\boldsymbol{\lambda}_{\boldsymbol{i}_{t,o}} = t\boldsymbol{\lambda}_{\boldsymbol{i}_{t,o}} \\
& \dot{\boldsymbol{g}}_{\boldsymbol{i}_{t,o}}(\boldsymbol{q}(t),\boldsymbol{\varphi}(t)) + \boldsymbol{\lambda}_{\boldsymbol{i}_{t,o}}\underbrace{(\gamma_i^{-1} - t)}_{\geq 0} = \boldsymbol{0} \\
\Rightarrow \quad & \|\boldsymbol{\lambda}_{\boldsymbol{i}_{t,o}}\|_2(\gamma_i^{-1} - t) = \|\dot{\boldsymbol{g}}_{\boldsymbol{i}_{t,o}}(\boldsymbol{q}(t),\boldsymbol{\varphi}(t))\|_2 \\
& t = -\frac{\|\dot{\boldsymbol{g}}_{\boldsymbol{i}_{t,o}}(\boldsymbol{q}(t),\boldsymbol{\varphi}(t))\|_2}{\|\boldsymbol{\lambda}_{\boldsymbol{i}_{t,o}}\|_2} + \gamma_i^{-1} \\
\Rightarrow \quad & \boldsymbol{\lambda}_{\boldsymbol{i}_{t,o}} = -\frac{\dot{\boldsymbol{g}}_{\boldsymbol{i}_{t,o}}(\boldsymbol{q}(t),\boldsymbol{\varphi}(t))}{\|\dot{\boldsymbol{g}}_{\boldsymbol{i}_{t,o}}(\boldsymbol{q}(t),\boldsymbol{\varphi}(t))\|_2}\|\boldsymbol{\lambda}_{\boldsymbol{i}_{t,o}}\|_2 \\
& \quad = -\frac{\dot{\boldsymbol{g}}_{\boldsymbol{i}_{t,o}}(\boldsymbol{q}(t),\boldsymbol{\varphi}(t))}{\|\dot{\boldsymbol{g}}_{\boldsymbol{i}_{t,o}}(\boldsymbol{q}(t),\boldsymbol{\varphi}(t))\|_2}\mu_i(\boldsymbol{z}(t))\lambda_{i_n}
\end{aligned}$$

Thus we recovered Eq. (13b). Eq. (13a) can be readily recovered by observing that if the frictional force is not at its limits then the VI requires that $\dot{\boldsymbol{g}}_{\boldsymbol{i}_{t,o}}(\boldsymbol{q}(t), \boldsymbol{\varphi}(t)) + \gamma_i^{-1}\boldsymbol{\lambda}_{\boldsymbol{i}_{t,o}} = \mathbf{0}$.

To merge Eq. (12) and Eq. (14) into a single VI, we rescale Eq. (12) by $\frac{1}{\delta t}$ for convenience and introduce an auxiliary vector $\boldsymbol{f}$, where

$$\boldsymbol{f}_{\boldsymbol{i}}(\boldsymbol{q}(t), \boldsymbol{\varphi}(t), \boldsymbol{\lambda}) = \begin{pmatrix} \frac{1}{\delta t} g_{i_n}(\boldsymbol{q}(t), \boldsymbol{\varphi}(t)) + \frac{k_i^{-1}}{\delta t}\lambda_{i_n} \\ \dot{\boldsymbol{g}}_{\boldsymbol{i}_{t,o}}(\boldsymbol{q}(t), \boldsymbol{\varphi}(t)) + \gamma_i^{-1}\boldsymbol{\lambda}_{\boldsymbol{i}_{t,o}} \end{pmatrix}. \tag{16}$$

Then rescaled Eq. (12) and Eq. (14) together form, for a system with frictional contacts only, the VI

$$\boldsymbol{\lambda} \in \mathcal{C} = \prod_i [0;\infty) \times \mathcal{S}(\mu_i(\boldsymbol{z}(t))\bar{\lambda}_{i_n}), \quad \langle \boldsymbol{f}(\boldsymbol{q}(t), \boldsymbol{\varphi}(t), \boldsymbol{\lambda}), \boldsymbol{y} - \boldsymbol{\lambda} \rangle \geq 0, \quad \forall \boldsymbol{y} \in \mathcal{C}, \tag{17}$$

where the constraint set $\mathcal{C}$ is the Cartesian product of friction cylinders. In order for Eq. (17) to be accurate a reasonable approximation of the normal reactions must be available or a fixed-point iteration can be wrapped around. Alternatively, the constraint set can be chosen to be the Cartesian product of friction cones $\mathcal{F}_{\boldsymbol{i}} = \left\{ \boldsymbol{\lambda}_{\boldsymbol{i}} \in \mathbb{R}^3 \;\middle|\; \lambda_{i_n} \geq 0, \|\boldsymbol{\lambda}_{\boldsymbol{i}_{t,o}}\|_2 \leq \mu_i(\boldsymbol{z}(t))\lambda_{i_n} \right\}$ as the Coulomb friction suggests:

$$\boldsymbol{\lambda} \in \mathcal{F} = \prod_i \mathcal{F}_{\boldsymbol{i}}, \quad \langle \boldsymbol{f}(\boldsymbol{q}(t), \boldsymbol{\varphi}(t), \boldsymbol{\lambda}), \boldsymbol{y} - \boldsymbol{\lambda} \rangle \geq 0, \quad \forall \boldsymbol{y} \in \mathcal{F}. \tag{18}$$

However, in this approach, as investigated e.g. in [7], the complementarity of λ_{i_n} and $g_{i_n}(\boldsymbol{q}(t), \boldsymbol{\varphi}(t))$ is lost for dynamic closed contacts. The unintentional vertical motions can be interpreted as originating from surface asperities though.

To discretize Eq. (18) we first discretize Eq. (16)

$$\begin{aligned} \tilde{\boldsymbol{f}}_{\boldsymbol{i}}(\tilde{\boldsymbol{\lambda}}) &= \begin{pmatrix} \frac{1}{\delta t} g_{i_n}(\tilde{\boldsymbol{q}}', \tilde{\boldsymbol{\varphi}}') + \frac{k_i^{-1}}{\delta t}\tilde{\lambda}_{i_n} \\ \mathbf{J}_{\boldsymbol{i}_{t,o}*}(\tilde{\boldsymbol{q}}, \tilde{\boldsymbol{\varphi}}) \begin{pmatrix} \tilde{\boldsymbol{v}}' \\ \tilde{\boldsymbol{\omega}}' \end{pmatrix} + \gamma_i^{-1}\tilde{\boldsymbol{\lambda}}_{\boldsymbol{i}_{t,o}} \end{pmatrix} \\ &= \frac{1}{\delta t}\boldsymbol{g}_{\boldsymbol{i}}(\tilde{\boldsymbol{q}}, \tilde{\boldsymbol{\varphi}}) + \left(\mathbf{J}_{\boldsymbol{i}*}(\tilde{\boldsymbol{q}}, \tilde{\boldsymbol{\varphi}})\tilde{\mathbf{M}}^{-1}\mathbf{J}(\tilde{\boldsymbol{q}}, \tilde{\boldsymbol{\varphi}})^T + \frac{1}{\delta t}\operatorname{diag}(\boldsymbol{h})_{\boldsymbol{i}*}^{-1} \right) \delta t \tilde{\boldsymbol{\lambda}} \\ &\quad + \mathbf{J}_{\boldsymbol{i}*}(\tilde{\boldsymbol{q}}, \tilde{\boldsymbol{\varphi}}) \left(\begin{pmatrix} \tilde{\boldsymbol{v}} \\ \tilde{\boldsymbol{\omega}} \end{pmatrix} + \delta t \tilde{\mathbf{M}}^{-1} \begin{pmatrix} \tilde{\boldsymbol{f}} \\ \tilde{\boldsymbol{\tau}} - \tilde{\boldsymbol{\omega}} \times \tilde{\hat{\mathbf{I}}}\tilde{\boldsymbol{\omega}} \end{pmatrix} \right), \end{aligned} \tag{19}$$

where $\boldsymbol{g}_{\boldsymbol{i}_{t,o}}(\tilde{\boldsymbol{q}}, \tilde{\boldsymbol{\varphi}})$ is assumed to be $\mathbf{0}$ and $\boldsymbol{h}_{\boldsymbol{i}} = (k_i\delta t, \gamma_i, \gamma_i)^T$ and thus obtain

$$\tilde{\boldsymbol{\lambda}} \in \tilde{\mathcal{F}} = \prod_i \tilde{\mathcal{F}}_{\boldsymbol{i}}, \quad \langle \tilde{\boldsymbol{f}}(\tilde{\boldsymbol{\lambda}}), \boldsymbol{y} - \tilde{\boldsymbol{\lambda}} \rangle \geq 0, \quad \forall \boldsymbol{y} \in \tilde{\mathcal{F}}, \tag{20}$$

where $\tilde{\mathcal{F}}_{\boldsymbol{i}} = \left\{ \tilde{\boldsymbol{\lambda}}_{\boldsymbol{i}} \in \mathbb{R}^3 \;\middle|\; \tilde{\lambda}_{i_n} \geq 0, \|\tilde{\boldsymbol{\lambda}}_{\boldsymbol{i}_{t,o}}\|_2 \leq \mu_i(\tilde{\boldsymbol{z}})\tilde{\lambda}_{i_n} \right\}$. Inserting Eq. (2b) and Eq. (5) into Eq. (20) yields, for a system with frictional contacts only, the affine variational inequality problem (AVI)

$$\begin{aligned} \tilde{\boldsymbol{\lambda}} \in \tilde{\mathcal{F}}, \quad \Bigg\langle & \frac{1}{\delta t}\boldsymbol{g}(\tilde{\boldsymbol{q}}, \tilde{\boldsymbol{\varphi}}) + \left(\mathbf{J}(\tilde{\boldsymbol{q}}, \tilde{\boldsymbol{\varphi}})\tilde{\mathbf{M}}^{-1}\mathbf{J}(\tilde{\boldsymbol{q}}, \tilde{\boldsymbol{\varphi}})^T + \frac{1}{\delta t}\operatorname{diag}(\boldsymbol{h})^{-1} \right) \delta t \tilde{\boldsymbol{\lambda}} \\ & + \mathbf{J}(\tilde{\boldsymbol{q}}, \tilde{\boldsymbol{\varphi}}) \left(\begin{pmatrix} \tilde{\boldsymbol{v}} \\ \tilde{\boldsymbol{\omega}} \end{pmatrix} + \delta t \tilde{\mathbf{M}}^{-1} \begin{pmatrix} \tilde{\boldsymbol{f}} \\ \tilde{\boldsymbol{\tau}} - \tilde{\boldsymbol{\omega}} \times \tilde{\hat{\mathbf{I}}}\tilde{\boldsymbol{\omega}} \end{pmatrix} \right), \boldsymbol{y} - \tilde{\boldsymbol{\lambda}} \Bigg\rangle \geq 0, \quad \forall \boldsymbol{y} \in \tilde{\mathcal{F}} \end{aligned} \tag{21}$$

as time step problems. Neglecting the current gap $\boldsymbol{g}(\tilde{\boldsymbol{q}}, \tilde{\boldsymbol{\varphi}})$ and introducing a regularization parameter s, this AVI is of the type

$$\boldsymbol{x} \in \delta t\tilde{\mathcal{F}}, \quad \langle (\mathbf{A}^T\mathbf{A} + s\mathbf{D})\boldsymbol{x} - \mathbf{A}^T\boldsymbol{b}, \boldsymbol{y} - \boldsymbol{x}\rangle \geq 0, \quad \forall \boldsymbol{y} \in \delta t\tilde{\mathcal{F}}. \tag{22}$$

For the case $\mathbf{D} = \mathbf{I}$ it was proved in [8] that the solution of such a regularized VI converges to a minimum norm solution of the unregularized VI (Eq. (22) with $s = 0$) as $s \to 0$. By substituting $\boldsymbol{x} = \mathbf{D}^{-\frac{1}{2}}\boldsymbol{z}$ one can show that the solution converges for $\mathbf{D}$ symmetric positive-definite (SPD) to a weighted minimum norm solution.

$$\begin{aligned}
& \boldsymbol{x} \in \delta t\tilde{\mathcal{F}}, \quad ((\mathbf{A}^T\mathbf{A} + s\mathbf{D})\boldsymbol{x} - \mathbf{A}^T\boldsymbol{b})^T(\boldsymbol{y} - \boldsymbol{x}) \geq 0, \quad \forall \boldsymbol{y} \in \delta t\tilde{\mathcal{F}} \\
\overset{[9]}{\leftrightarrow} & \min_{\boldsymbol{x}\in\delta t\tilde{\mathcal{F}}} \frac{1}{2}\boldsymbol{x}^T(\mathbf{A}^T\mathbf{A} + s\mathbf{D})\boldsymbol{x} - \boldsymbol{x}^T\mathbf{A}^T\boldsymbol{b} \\
\overset{\boldsymbol{x}=\mathbf{D}^{-\frac{1}{2}}\boldsymbol{z}}{\leftrightarrow} & \min_{\mathbf{D}^{-\frac{1}{2}}\boldsymbol{z}\in\delta t\tilde{\mathcal{F}}} \frac{1}{2}\boldsymbol{z}^T\mathbf{D}^{-\frac{T}{2}}(\mathbf{A}^T\mathbf{A} + s\mathbf{D})\mathbf{D}^{-\frac{1}{2}}\boldsymbol{z} - \boldsymbol{z}^T\mathbf{D}^{-\frac{T}{2}}\mathbf{A}^T\boldsymbol{b} \\
\overset{\mathbf{A}\mathbf{D}^{-\frac{1}{2}}=\hat{\mathbf{A}}}{\leftrightarrow} & \min_{\boldsymbol{z}\in\delta t\mathbf{D}^{\frac{1}{2}}\tilde{\mathcal{F}}} \frac{1}{2}\boldsymbol{z}^T(\hat{\mathbf{A}}^T\hat{\mathbf{A}} + s\mathbf{I})\boldsymbol{z} - \boldsymbol{z}^T\hat{\mathbf{A}}^T\boldsymbol{b} \\
\overset{[9]}{\leftrightarrow} & \boldsymbol{z} \in \delta t\mathbf{D}^{\frac{1}{2}}\tilde{\mathcal{F}}, \quad ((\hat{\mathbf{A}}^T\hat{\mathbf{A}} + s\mathbf{I})\boldsymbol{z} - \hat{\mathbf{A}}^T\boldsymbol{b})^T(\boldsymbol{y} - \boldsymbol{z}) \geq 0, \quad \forall \boldsymbol{y} \in \delta t\mathbf{D}^{\frac{1}{2}}\tilde{\mathcal{F}}
\end{aligned} \tag{23}$$

According to [8] and since $\delta t\mathbf{D}^{\frac{1}{2}}\tilde{\mathcal{F}}$ is still a nonempty closed convex set, for $s \to 0$ the solution $\boldsymbol{z}$ converges to the minimum norm solution of the unregularized VI (Eq. (22) with $s = 0$) and thus when substituting back one obtains that $\boldsymbol{x}$ converges to the weighted minimum norm solution

$$\min_{\boldsymbol{z}\in\delta t\mathbf{D}^{\frac{1}{2}}\tilde{\mathcal{F}},\ \langle\hat{\mathbf{A}}^T\hat{\mathbf{A}}\boldsymbol{z}-\hat{\mathbf{A}}^T\boldsymbol{b},\boldsymbol{y}-\boldsymbol{z}\rangle\geq 0,\ \forall\boldsymbol{y}\in\delta t\mathbf{D}^{\frac{1}{2}}\tilde{\mathcal{F}}} \boldsymbol{z}^T\boldsymbol{z} = \min_{\boldsymbol{x}\in\delta t\tilde{\mathcal{F}},\ \langle\mathbf{A}^T\mathbf{A}\boldsymbol{x}-\mathbf{A}^T\boldsymbol{b},\boldsymbol{y}-\boldsymbol{x}\rangle\geq 0,\ \forall\boldsymbol{y}\in\delta t\tilde{\mathcal{F}}} \boldsymbol{x}^T\mathbf{D}\boldsymbol{x}. \tag{24}$$

Note that this proof includes linear complementarity problems (LCP) as a special case for $\tilde{\mathcal{F}} = \prod_i [0; \infty)$.

4 CONCLUSION

The presented regularization models frictionless unilateral contacts taking impacts into consideration and also handles a certain class of dry frictional contacts including a modified Coulomb friction. The discretization of these regularized time step problems leads to a variational inequality problem, which is equivalent to formulations by Anitescu et al [7, 10] in the stiff limit. The limiting process identifies numerical damping as the reason for the inelasticity of the contacts and characterizes the regularized solution of the unregularized problem as the unique solution of weighted minimum norm. The limiting process in itself suggests a crude numerical treatment by iteratively reducing the influence of the regularization matrix [8]. However, more work is required to select efficient numerical schemes. Aside from that a rigorous analysis for true Coulomb friction is still to be done and the modeling of the damping in the unilateral contact should be reconsidered. Instead of being introduced artificially by numerical damping, dampers should be included in the model right from the start. This, however, would also require a change of the discretization scheme.

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