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Some Remarks on a Collocation Method for First Kind Integral Equations

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SOME REMARKS ON A COLLOCATION METHOD FOR FIRST KIND INTEGRAL EQUATIONS *

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Abstract. In [4] the authors proposed a collocation algorithm for approximating the minimal norm least-squares solution of first kind integral equations with continuous reproducing kernel. The convergence of the approximate solutions sequence is proved under the assumption that all the sets of collocation functions are linearly independent. In this paper we replace the above assumption by a weaker one and prove that the corresponding sequence of approximations still converges to the minimal norm solution of the initial equation.

1 Introduction

We shall use through the whole paper the notations and results from [4] (including those refering to the construction and properties of reproducing kernel Hilbert spaces (RKHS)). For a linear operator A, $N(A)$ and $R(A)$ will denote the null space and range of A, respectively. Moreover, by $span\{v_1, \dots, v_m\}$ we shall denote the linear subspace spanned by the family $\{v_1, \dots, v_m\}$ from a given vector space. L^2 will be the real Hilbert space $L^2([0,1])$ with the scalar product $\langle \cdot, \cdot \rangle$ and the associated norm $\| \cdot \|$, respectively. Let $K : L^2 \longrightarrow L^2$ be the integral operator

$$Kx(t) = \int_0^1 k(t,s)x(s)ds, \tag{1}$$

with the kernel $k : [0,1] \times [0,1] \longrightarrow I\!R$ and $y \in L^2$ a given function. We consider the first kind integral equation: find $x \in L^2$ such that

$$Kx(t) = y(t), \forall t \in [0,1]. \tag{2}$$

If $T_m = \{t_1, \dots, t_m\}$ is the set of collocation points in $[0,1]$, we consider the collocation formulation of (2): find $x \in L^2$ such that

$$Kx(t_i) = y(t_i), \ \forall i = 1, \dots, m. \tag{3}$$

For $t_i \in T_m$ we define $k_{t_i} : [0,1] \longrightarrow I\!R$ and $\tilde{y_i}$ by

$$k_{t_i}(s) = k(t_i, s), \ \forall s \in [0,1], \ \tilde{y_i} = y(t_i), \ i = 1, \dots, m. \tag{4}$$

Then, the equation (3) can be written as

$$C_m x = \tilde{y}, \tag{5}$$

where $\tilde{y} \in I\!R^m$ and $C_m : L^2 \longrightarrow I\!R^m$ are defined by

$$C_m z = \begin{bmatrix} \langle k_{t_1}, z \rangle \\ . \\ . \\ . \\ \langle k_{t_m}, z \rangle \end{bmatrix}, \ \tilde{y} = \begin{bmatrix} \tilde{y_1} \\ . \\ . \\ . \\ \tilde{y_m} \end{bmatrix}. \tag{6}$$

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If

$$y \in R(K), \tag{7}$$

let x^{LS} be the minimal norm least-squares solution of (2) and x_m^{LS} the similar one for (5), given by

$$x^{LS} = K^+ y, \quad x_m^{LS} = C_m^+ \tilde{y}, \tag{8}$$

where by A^+ we shall denote the Moore-Penrose pseudoinverse of a linear operator A (see e.g. [3]). The following assumption was used in [4].
Assumption A1. *The set of collocation functions defined by the collocation points in T_m, i.e.*

$$\{k_{t_i}, i = 1, \dots, m\} \tag{9}$$

is linearly independent, $\forall \;\; m \geq 1$.
Let Δ_m be defined

$$\Delta_m = \sup_{t \in [0,1]} \left(\inf_{t_i \in T_m} |(t - t_i)| \right). \tag{10}$$

The following result is proved in [4] (Theorem 3.1, partial statement).
Theorem 1. *Under the above assumption* **A1**, *if* (7) *holds, the reproducing kernel $Q(t, t')$, defined by*

$$Q(t, t') = \int_0^1 k(t, s) k(t', s) ds, \;\; t, t' \in [0, 1] \tag{11}$$

is continuous and

$$\lim_{m \longrightarrow \infty} \Delta_m = 0, \tag{12}$$

then

$$\lim_{\Delta_m \longrightarrow 0} \parallel x_m^{LS} - x^{LS} \parallel = 0. \tag{13}$$

2 Convergence proof for $\mathbf{y} \in \mathbf{R}(\mathbf{K})$

We shall firstly replace the assumption **A1** by a weaker one. In this sense, we denote by

$$X_m = span\{k_t, t \in T_m\} \tag{14}$$

the finite dimensional vector subspace of collocation functions defined by the set T_m (see (4)) and by $dim(X_m)$ its dimension.
Assumption WA1. *It exists a sequence of positive integers* $0 < m_1 < m_2 < \dots < m_p < m_{p+1} < \dots$ *such that*

$$dim(X_{m_p}) \;\; < \;\; dim(X_{m_{p+1}}), \;\; \forall p \geq 1. \tag{15}$$

Remark 1. *From the definition of $T_m = \{t_1, \dots, t_m\} \subset [0, 1]$ (see also [4]) we have $T_m \subset T_{m+1}, \forall m \geq 1$, which means that $X_m \subset X_{m+1}, \forall m \geq 1$. Thus, the above assumption* **WA1** *tells us that the number of linearly independent functions k_t in the subspaces X_m tends to infinity, but not all the functions in each X_m are linearly independent, as in the assumption* **A1**.
Remark 2. *Assumption* **WA1** *together with* (12) *ensures the fact that the set $\{Q_t, t \in \bigcup_{m \geq 1} T_m\}$ is dense in the reproducing kernel space $\mathcal{K}_Q$ with respect to the $\langle \cdot, \cdot \rangle_Q$ (see [4]).*
Remark 3. *In* [4] *it is proved that, under the assumption* **A1** *the minimal norm solution x_m^{LS} from* (8) *is given by*

$$x_m^{LS} = (\tilde{y}_1, \dots, \tilde{y}_m) Q_m^{-1} \begin{bmatrix} k_{t_1} \\ \cdot \\ \cdot \\ \cdot \\ k_{t_m} \end{bmatrix}, \tag{16}$$

where Q_m is the Gram matrix of the linearly independent system $\{k_t, t \in T_m\}$.
Under our weaker assumption **WA1**, the set $\{k_t, t \in T_m\}$ is no more linearly independent, thus the associated Gram matrix Q_m, defined by

$$(Q_m)_{ij} = \langle k_{t_i}, k_{t_j} \rangle, \ \forall i, j = 1, \dots, m \tag{17}$$

is no more invertible. Thus, in order to compute x_m^{LS} from (8), we need some auxiliary results which shall be presented in what follows.
Lemma 1. *Let $(H, (\cdot, \cdot))$ be a real Hilbert space, $x \in H$ an arbitrary element, V_m a finite dimensional subspace $V_m = span\{v_1, \dots, v_m\} \subset H$, the $m \times m$ matrix G_m and $b \in I\!R^m$ defined by*

$$(G_m)_{ij} = (v_j, v_i), \ b_i = (x, v_i). \tag{18}$$

Then, the orthogonal projection of x onto V_m, denoted by $P_{V_m}(x)$ is given by

$$P_{V_m}(x) = \sum_{j=1}^{m} \alpha_j^{LS} v_j, \tag{19}$$

where $\alpha^{LS} = (\alpha_1^{LS}, \dots, \alpha_m^{LS})^t \in I\!R^m$ is the minimal norm solution of the system

$$G_m \alpha = b. \tag{20}$$

Proof. It is clear that the system (20) is consistent and $P_{V_m}(x)$ is of the form (19). By using the well known characterization of projection, i.e. $x - P_{V_m}(x) \perp V_m$, we obtain that the coefficients vector α^{LS} from (19) must be a solution of (20). Because the system $\{v_1, \cdots, v_m\}$ is arbitrary, the Gram matrix G_m is symmetric and positive semi-definite, but not more invertible. Thus, the set of all solutions of (20) is given by (see e.g. [1])

$$S(G_m; b) = \{\alpha^{LS} + w, w \in N(G_m)\}, \tag{21}$$

where

$$\alpha^{LS} = G_m^+ b \tag{22}$$

is the minimal norm one. If $\alpha = \alpha^{LS} + w \in S(G_m; b)$ is an arbitrary solution, we have

$$\sum_{j=1}^{m} \alpha_j v_j = \alpha^t \begin{bmatrix} v_1 \\ . \\ . \\ . \\ v_m \end{bmatrix} = (\alpha^{LS})^t \begin{bmatrix} v_1 \\ . \\ . \\ . \\ v_m \end{bmatrix} + w^t \begin{bmatrix} v_1 \\ . \\ . \\ . \\ v_m \end{bmatrix}. \tag{23}$$

Let $y = w^t \begin{bmatrix} v_1 \\ . \\ . \\ . \\ v_m \end{bmatrix} = \sum_{j=1}^m w_j v_j$. Because $w \in N(G_m)$, we obtain

$$\sum_{j=1}^{m} (w_j v_j, v_i) = (\sum_{j=1}^{m} w_j v_j, v_i) = (y, v_i) = 0, \ \forall i = 1, \dots, m. \tag{24}$$

But $y \in V_m$, thus using (24) we obtain $y = 0$, which means that

$$\forall \ \ \alpha \in S(G_m; b), \ \sum_{j=1}^{m} \alpha_j v_j = \sum_{j=1}^{m} \alpha_j^{LS} v_j \tag{25}$$

and the proof is complete.
Lemma 2. *The minimal norm solution of the system* (5) *belongs to the subspace X_m from* (14).
Proof. Let $C_m^* : I\!R^m \longrightarrow L^2$ be the adjoint of C_m from (6), defined by

$$\langle C_m v, z \rangle_2 = \langle v, C_m^* z \rangle, \ \ \forall \ \ z \in I\!R^m, \ \ v \in L^2, \tag{26}$$

where by $\langle \cdot, \cdot \rangle_2$ we denoted the Euclidean scalar product in $I\!R^m$. By a simple computation using (6) and (26), we obtain that

$$R(C_m^*) = X_m. \tag{27}$$

Let $\tilde{C}_m : R(C_m^*) \longrightarrow R(C_m)$ be the restriction of C_m to the subspace $R(C_m^*) \subset L^2$. Then (see e.g. [1]) $\tilde{C}_m$ is bijective and the Moore-Penrose pseudoinverse of C_m is defined by $C_m^+ : I\!R^m \longrightarrow R(C_m^*)$,

$$C_m^+ z = \tilde{C}_m^{-1}(z_1), \quad \forall \quad z = z_1 \oplus z_2 \in R(C_m) \oplus R(C_m)^\perp = I\!R^m. \tag{28}$$

Thus, for x_m^{LS} from (8) we get (also using (28))

$$x_m^{LS} = C_m^+ \cdot \tilde{y} \in R(C_m^*) = X_m$$

and the proof is complete.
According to the above Lemma 2, let $x_m^{LS} \in X_m$ be given by (see (14))

$$x_m^{LS} = \sum_{j=1}^{m} \alpha_j \cdot k_{t_j}. \tag{29}$$

From (5) and (29) we obtain that the vector $\alpha = (\alpha_1, \dots, \alpha_m)^t \in I\!R^m$ must be a solution of the (consistent) system

$$Q_m \cdot \alpha = \tilde{y}, \tag{30}$$

with Q_m from (17). But, as in the proof of Lemma 1 before, we show that for any solution $\alpha = \alpha^{LS} + w$ of (30) (with $\alpha^{LS} = Q_m^+ \tilde{y}$, $w \in N(Q_m)$) we obtain $\sum_{j=1}^m \alpha_j k_{t_j} = \sum_{j=1}^m \alpha_j^{LS} k_{t_j}$, which means that x_m^{LS} from (29) will be given by (see (16))

$$x_m^{LS} = (\tilde{y}_1, \dots, \tilde{y}_m) Q_m^+ \begin{bmatrix} k_{t_1} \\ \cdot \\ \cdot \\ \cdot \\ k_{t_m} \end{bmatrix}. \tag{31}$$

Then, we obtain as in [4] that

$$K x_m^{LS} = (\tilde{y}_1, \dots, \tilde{y}_m) Q_m^+ \begin{bmatrix} Q_{t_1} \\ \cdot \\ \cdot \\ \cdot \\ Q_{t_m} \end{bmatrix}, \tag{32}$$

where the functions Q_t, with $t \in [0, 1]$ are defined with the reproducing kernel $Q(t, t')$ by

$$Q_t(s) = Q(t, s), \quad \forall \quad s \in [0, 1]. \tag{33}$$

Moreover, the following properties hold (see (17) and [4]):

$$\langle Q_{t_i}, Q_{t_j} \rangle_Q = \langle k_{t_i}, k_{t_j} \rangle = (Q_m)_{ij}, \tag{34}$$

$$\langle Q_t, z \rangle_Q = z(t), \quad \forall \quad z \in \mathcal{K}_Q, \quad t \in [0, 1], \tag{35}$$

where $\mathcal{K}_Q$ is $R(K)$ organized as a Hilbert space with reproducing kernel $Q(t, t')$. Then, as in the proof of Theorem 3.1 from [4] we obtain that

$$\parallel x_m^{LS} - x^{LS} \parallel = \parallel K x_m^{LS} - K x^{LS} \parallel_Q = \parallel K x_m^{LS} - y \parallel_Q . \tag{36}$$

Thus (see also Remark 2 before), the only things that have to be proved are the following:

$$K x_m^{LS} = P_{T_m}(y), \tag{37}$$

where $P_{T_m}(y)$ is the projection of y in $\mathcal{K}_Q$ onto the vector subspace generated by the set $\{Q_t, t \in T_m\}$ and that

$$\lim_{\Delta_m \longrightarrow 0} \parallel y - P_{T_m}(y) \parallel_Q = 0. \tag{38}$$

For (37), by replacing in Lemma 1 $(H, \langle\cdot,\cdot\rangle)$ with $(\mathcal{K}_Q, \langle\cdot,\cdot\rangle_Q)$ and V_m with $span\{Q_t,\ t \in T_m\}$, we obtain that

$$P_{T_m}(y) = \sum_{j=1}^{m} \alpha_j^{LS} \cdot Q_{t_j}, \tag{39}$$

with $\alpha^{LS} \in I\!R^m$ the minimal norm solution of the system

$$\tilde{Q}_m \alpha^{LS} = \tilde{y} \tag{40}$$

and the $m \times m$ matrix $\tilde{Q}_m$ given by (see (17), (39), (34))

$$(\tilde{Q}_m)_{ij} = \langle Q_{t_j}, Q_{t_i}\rangle_Q = (Q_m)_{ij}. \tag{41}$$

But, from (39)-(41) and Lemma 1 it results that α^{LS} is the minimal norm solution of (30), i.e.

$$\alpha^{LS} = Q_m^+ \tilde{y}, \tag{42}$$

which together with (32) and (39) gives us (37).
For (38), let $\epsilon > 0$ be arbitrary fixed and $w_\epsilon \in span\{Q_t,\ t \in \bigcup_{m\geq 1} T_m\}$ such that (see Remark 2)

$$\parallel y - w_\epsilon \parallel_Q \ < \ \epsilon. \tag{43}$$

But

$$w_\epsilon = \sum_{l=1}^{p} \alpha_l \cdot Q_{t^l}, \quad t^l \in T_{m_l}, \quad l = 1, \ldots, p. \tag{44}$$

Thus, if we define $m_\epsilon \geq 1$ by

$$m_\epsilon = max\{m_1, \ldots, m_p\}, \tag{45}$$

then $t^l \in T_{m_\epsilon}$, $\forall l = 1, \ldots, p$, so

$$w_\epsilon \in T_{m_\epsilon}. \tag{46}$$

From (46) it results that

$$\parallel y - P_{T_{m_\epsilon}}(y) \parallel_Q \ \leq \ \parallel y - w_\epsilon \parallel_Q \ \leq \ \epsilon. \tag{47}$$

Now, if

$$m \geq m_\epsilon \tag{48}$$

we have that $T_{m_\epsilon} = \{t_1, \ldots, t_{m_\epsilon}\} \subset T_m = \{t_1, \ldots, t_m\}$ and thus

$$span\{Q_{t_1}, \ldots, Q_{t_{m_\epsilon}}\} \subset span\{Q_{t_1}, \ldots, Q_{t_m}\},$$

which means that

$$\parallel y - P_{T_m}(y) \parallel_Q \ \leq \ \parallel y - P_{T_{m_\epsilon}}(y) \parallel_Q \ \leq \ \epsilon. \tag{49}$$

Then, (38) directly holds from (47)-(49).
In this way we proved the following result.
Theorem 2. *Under the hypothesis of Theorem 1 and if the assumption* **A1** *is replaced by* **WA1***, then* (13) *holds.*

3 Convergence proof for $\mathbf{y} \in \mathbf{R}(\mathbf{K}) \oplus \mathbf{R}(\mathbf{K})^\perp$

In this case, in [4] the authors considered instead of (2), its normal equation

$$\tilde{Q}x = w, \tag{50}$$

where $\tilde{Q} = K^*K, w = K^*y$ and K^* the adjoint of K from (1). Because of the equality (see [3], [4])

$$\tilde{Q}^+ w = K^+ y, \tag{51}$$

it results that the equations (2) and (50) have the same minimal norm solution x^{LS} from (8). Then, if $S_m = \{s_1, \ldots, s_m\} \subset [0,1]$, is the set of collocation points, we replace (3) by the problem: find $x \in L^2$ such that

$$\sum_{i=1}^{m} \left(\tilde{Q}x(s_i) - w(s_i) \right)^2 = \min! \tag{52}$$

Let then $\tilde{q}_{s_i} : [0,1] \longrightarrow I\!R, \tilde{w} \in I\!R^m$ be given by (see (4))

$$\tilde{q}_{s_i}(t) = \tilde{Q}(s_i, t), \ t \in [0,1], \tilde{w} = (w(s_1), \ldots, w(s_m))^t. \tag{53}$$

If $Y_m = span\{\tilde{q}_{s_1}, \ldots, \tilde{q}_{s_m}\}$, we shall replace authors' assumption
" the set $\{\tilde{q}_{s_1}, \ldots, \tilde{q}_{s_m}\}$ is linearly independent $\forall m \geq 1$" by the following weaker one.
Assumption GWA1. *It exists a sequence of positive integers*
$0 < m_1 < m_2 < \cdots < m_p < m_{p+1} < \cdots$ *such that*

$$dim(Y_{m_p}) < dim(Y_{m_{p+1}}), \forall p \geq 1. \tag{54}$$

Then, if P_m is the $m \times m$ symmetric and positive semi-definite matrix defined by

$$(P_m)_{ij} = \langle \tilde{q}_{s_j}, \tilde{q}_{s_i} \rangle, \ \forall i, j = 1, \ldots, m, \tag{55}$$

as in Section 2 we can show that the minimal norm solution $\tilde{x}_m^{LS}$ of (52) is given by

$$\tilde{x}_m^{LS} = (\tilde{w}_1, \ldots, \tilde{w}_m) P_m^+ (\tilde{q}_{s_1}, \ldots, \tilde{q}_{s_m})^t. \tag{56}$$

We shall also suppose that for $\tilde{\Delta}_m$ defined by

$$\tilde{\Delta}_m = \sup_{s \in [0,1]} \left(\inf_{s_i \in S_m} |(s - s_i)| \right) \tag{57}$$

the following holds

$$\lim_{m \to \infty} \tilde{\Delta}_m = 0. \tag{58}$$

Then, by using similar arguments as those in the above Section 1 we can prove the following result (see Theorem 4.1 from [4]).
Theorem 3. *Under the assumption* **GWA1***, if $y \in R(K) \oplus R(K)^\perp$, the reproducing kernel*

$$P(s, s') = \int_0^1 \tilde{Q}(s, u) \tilde{Q}(s', u) du, \ \ s, s' \in [0,1]$$

is continuous and (57) *holds, then*

$$\lim_{\tilde{\Delta}_m \longrightarrow 0} \parallel \tilde{x}_m^{LS} - x^{LS} \parallel = 0. \tag{59}$$

Remark 4. *Concerning practical applications of the above result we refer at the paper* [2] *in which, in the framework of systems arising from the discretization of boundary value problems, the author extends the initial (linearly independent) set of approximation functions, by some linear combination of them in order to improve the condition number of the system matrix. Work is in progress for applying this technique to the above described collocation algorithm.*

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