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Iterative Regularized Solution of Symmetric and Positive Semi-Definite Linear Complementarity Problems

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Abstract

In this report an iterative method from the theory of maximal monotone operators is transfered into the context of linear complementarity problems and numerical tests are performed on contact problems from the field of rigid multibody dynamics.

1 Introduction

Time-step problems arising in rigid multibody dynamics often exhibit large nullspaces due to redundant constraints. Abandoning rigidity in favor for flexible bodies removes this nullspace at the cost of the introduction of a large number of degrees of freedom. Thus computational feasibility might dictate to stick to the rigidity idealization. Instead of replacing bodies by a complete continuum mechanics based model one can introduce bodies deforming solely in the contact/joint region. However, such systems typically exhibit overlap since the amount of overlap determines the contact force. When increasing the stiffness of the contact regions one can show that the contact reaction converges towards a weighted minimum norm solution of the truly rigid problem [5]. If the multibody system is connected only by bilateral joints then the rigid time-step problems are linear systems of equations (LSEs) having symmetric positive semi-definite (SPSD) system matrices. Determining weighted minimum norm solutions of such LSEs corresponds to solving linear least squares (LLS) problems. Sec. 2 reviews an iterative method introduced in [5] to solve such problems. However, when dealing with contacts in a multibody system then the time step problems turn into linear complementarity problems (LCPs) with SPSD system matrices. Again the limiting process demands a weighted minimum norm solution of the LCP as investigated in [6]. Sec. 3 proposes an iterative method to compute minimum norm solutions of SPSD LCPs (monotone LCPs) borrowing a method for finding zeros of maximal monotone operators.

2 Iterative regularized solutions of weighted least squares problems

Let us consider the linear least squares problem

$$\|Ax^* - b\| = \min\{\|Ax - b\|, x \in \mathbb{R}^n\}, \tag{1}$$

where A is an $m \times n$ matrix, $b \in \mathbb{R}^m$ and denote by $LSS(A; b)$ the set of its solutions and by x_{ls} the minimum (Euclidean) norm one. It is well known that (1) can be expressed as a classical linear system through its

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associated normal equation

$$A^T A x^* - A^T b = 0. \tag{2}$$

Because $rank(A) = r$ is usually strictly less than $\min\{m, n\}$ we considered in [5] a regularized formulation of (2) of the form

$$A^T A x^*(s) - A^T b = -sDx^*(s), \tag{3}$$

where D is an SPD matrix, $s \in I\!R, s > 0$ and $x^*(s)$ its unique solution. Let us also consider the weighted LLS problem

$$\|D^{\frac{1}{2}} x^*_D\|^2 = \min_{x \in I\!R^n, s.t. \|Ax-b\|=\min!} \|D^{\frac{1}{2}} x\|^2, \tag{4}$$

with the unique solution $x^*_D \in I\!R^n$, i.e. $x^*_D \in LSS(A; b)$ is the unique solution for which $\|D^{\frac{1}{2}} x^*_D\|$ is minimum. In [6] we proved the following result.

Proposition 1. *With the above notations we have*

$$\lim_{s \to 0} x^*(s) = x^*_D. \tag{5}$$

For the efficient numerical solution of the regularized normal equation (3) we proposed in [5] the iteration

$$x^0 \in I\!R^n, (A^T A + sD)x^{k+1} = sDx^k + A^T b, \quad k \geq 0. \tag{6}$$

Note. The convergence of the iteration (6) was analysed in [7] in the case $D = I$ and $A^T A$ invertible, and in [3] for $D = I$, but a general $m \times n$ matrix A. In [5] we generalized these results for an arbitrary SPD matrix D, as described below.

Proposition 2. *The sequence* $\left(x^k\right)_{k \geq 0}$ *generated with the iteration* (6) *with* $x^0 = 0$ *and* $s > 0$ *fixed arbitrary, converges to the unique solution* x^*_D *of the weighted least squares problem* (4).

3 Iterative regularized solutions of linear complementarity problems

For A and b as in section 2 we consider the LCP(q, M) given by

$$q + Mz \geq 0 \perp z \geq 0, \text{ with } M = A^T A, \quad q = -A^T b. \tag{7}$$

In [2, page 395] the following (splitting based) algorithm is proposed for the numerical solution of (7):

Algorithm 3.1. *([2], Alg. 5.2.1)*
Let

$$M = B + C \tag{8}$$

be a splitting of M *from* (7).

Step 0. *Initialization. Let* $z^0 \in I\!R^n, z^0 \geq 0$*, set* $\nu = 0$

Step 1. *General iteration. Given* $z^\nu \geq 0$*, solve* LCP(q^ν, B)*, where*

$$q^\nu = q + Cz^\nu, \tag{9}$$

and let $z^{\nu+1}$ *be an arbitrary solution.*

Step 2. *Test for termination. If* $z^{\nu+1}$ *satisfies a prescribed stopping rule, terminate. Otherwise, return to* **Step 1** *with* ν *replaced by* $\nu + 1$.

Theorem 3.1. *([2], Th. 5.6.1)*
Let M, q *be those from* (7) *and* B *such that* $B - M$ *is symmetric and positive definite (SPD). Then, for* $z^0 \geq 0$ *an arbitrary initial approximation, the sequence* $(z^\nu)_{\nu \geq 0}$ *produced by the Alg. 3.1 converges to* some *element from* SOL(q, M).

Remark 1. *Unfortunately, there is no information in the proof of the above theorem in [2] which gives us details about the limit of the sequence* $(z^\nu)_{\nu\geq 0}$.

We considered in [6] a regularized version of the LCP (7) of the form

$$A^TAz - A^Tb \geq -sDz \quad \perp \quad z \geq 0. \tag{10}$$

Because the matrix $A^TA + sD$ is SPD the problem (10) has a unique solution, denoted in what follows by $z(s)$. According to Th. 5.6.2 in [2], under the assumption that $D = I$, the sequence of solutions $(z(s_\nu))_{\nu\geq 0}$, for a sequence of positive scalars $\{s_\nu\} \to 0$, converges to the minimum (Euclidean) norm solution of the unregularized LCP (7). By substituting $z = D^{-\frac{1}{2}}y$ and restricting D to a positive and diagonal matrix one can easily show that the sequence then converges to the weighted minimum norm solution z_{wmn}

$$\|D^{\frac{1}{2}}z_{wmn}\|_2^2 = \min_{z \text{ s.t. } A^TAz - A^Tb\geq 0 \;\perp\; z\geq 0} z^TDz. \tag{11}$$

Moreover, in [6] we proved the following generalization of Th. 5.6.2 in [2].

Theorem 3.2. *Let A be rectangular, D be a symmetric positive-definite matrix and let $\{s_\nu\}$ be a decreasing sequence of positive scalars with $s_\nu \to 0$. For each ν, let $z(s_\nu)$ be the unique solution of the LCP*

$$(A^TA + s_\nu D)z(s_\nu) - A^Tb \geq 0 \perp z(s_\nu) \geq 0. \tag{12}$$

Then the sequence $\{z(s_\nu)\}$ converges to the weighted minimum D-norm solution z_{wmn}

$$\|D^{\frac{1}{2}}z_{wmn}\|_2^2 = \min_{z \text{ s.t. } A^TAz - A^Tb\geq 0 \;\perp\; z\geq 0} z^TDz. \tag{13}$$

The algorithm that implicitly appears in Th. 3.2 requests in each iteration the numerical solution of an LCP of the form (10) with an iteration dependent matrix $A^TA + s_\nu D$. Moreover, if s_ν is small the condition number of the problem matrix will increase and will have a bad influence on the numerical solution. We can eliminate these unpleasant aspects by combining the regularization procedure (12) with the Alg. 3.1:

Algorithm 3.2. *For $M_{s_\nu} = A^TA + s_\nu D$ from (12) we consider the splitting*

$$M_{s_\nu} = B + C_{s_\nu}, \quad B = M + D, \quad C_{s_\nu} = (s_\nu - 1)D. \tag{14}$$

Step 0. *Initialization. Let $z^0 \in I\!R^n, z^0 \geq 0$, set $\nu = 0$*

Step 1. *General iteration. Given $z^\nu \geq 0$, solve* LCP(q^{s_ν}, B), *where*

$$q^{s_\nu} = q + C_{s_\nu} z^\nu, \tag{15}$$

and let $z^{\nu+1}$ be its unique solution.

Step 2. *Test for termination. If $z^{\nu+1}$ satisfies a prescribed stopping rule, terminate. Otherwise, return to* **Step 1** *with ν replaced by $\nu + 1$.*

Remark 2. *1. Because the sequence $(s_\nu)_{\nu\geq 0}$ is decreasing to 0 we may suppose that in* (14) *we have*

$$-1 < s_\nu - 1 < 0, \quad \forall \nu \geq 0. \tag{16}$$

2. In (each) Step 1 of the Alg. 3.2 we solve an LCP of which matrix B is independent on the iteration ν. Moreover, the corresponding regularized matrix M_{s_ν} makes the connection with the sequence of regularized problems from Th. 3.2.

In the above hypothesis, we can prove the following properties of the sequence $(z^\nu)_{\nu\geq 0}$ generated with the Alg. 3.2.

Proposition 3. *We have the inequalities*

$$\|z^{\nu+1} - z(s_\nu)\|_D \quad \leq \quad \|z^\nu - z(s_\nu)\|_D, \quad \forall \nu \geq 0, \tag{17}$$

and

$$\|z^{\nu+1} - z^\nu\|_D^2 \quad \leq \quad \|z^\nu - z(s_\nu)\|_D^2 - \|z^{\nu+1} - z(s_\nu)\|_D^2, \tag{18}$$

where $\|\cdot\|_D$ *is the energy norm on* $I\!R^n$ *defined by the SPD matrix* D.

Proof. Let F_ν be the SPD matrix defined by (see also (14) and (16))

$$F_\nu = B - M_{s_\nu} = -C_{s_\nu} = (1 - s_\nu)D. \tag{19}$$

According to the definitions of the vectors $z^{\nu+1}$ and $z(s_\nu)$ we have

$$z^{\nu+1} \geq 0, \quad q + C_{s_\nu} z^\nu + B z^{\nu+1} \geq 0, \quad \langle z^{\nu+1}, q + C_{s_\nu} z^\nu + B z^{\nu+1} \rangle = 0, \tag{20}$$

$$z(s_\nu) \geq 0, \quad q + (M + s_\nu D) z(s_\nu) \geq 0, \quad \langle z(s_\nu), q + (M + s_\nu D) z(s_\nu) \rangle = 0. \tag{21}$$

Thus, if we denote

$$w = q + C_{s_\nu} z^\nu + B z^{\nu+1}, \quad v = q + (M + s_\nu D) z(s_\nu), \tag{22}$$

we get from (20)-(22) that

$$\langle z^{\nu+1} - z(s_\nu), w - v \rangle = -\langle z^{\nu+1}, v \rangle - \langle z(s_\nu), w \rangle \leq 0.$$

Thus, from (19) and the fact that M_{s_ν} is SPD we successively obtain

$$0 \geq \langle z^{\nu+1} - z(s_\nu), w - v \rangle = 0 \geq \langle z^{\nu+1} - z(s_\nu), C_{s_\nu} z^\nu + B z^{\nu+1} - M_{s_\nu} z(s_\nu) \rangle =$$

$$\langle z^{\nu+1} - z(s_\nu), B(z^{\nu+1} - z(s_\nu)) + C_{s_\nu}(z^\nu - z(s_\nu)) \rangle = \langle z^{\nu+1} - z(s_\nu), F_{s_\nu}(z^{\nu+1} - z(s_\nu)) \rangle +$$

$$\langle z^{\nu+1} - z(s_\nu), M_{s_\nu}(z^{\nu+1} - z(s_\nu)) \rangle + \langle z^{\nu+1} - z(s_\nu), C_{s_\nu}(z^\nu - z(s_\nu)) \rangle \geq$$

$$\langle z^{\nu+1} - z(s_\nu), F_{s_\nu}(z^{\nu+1} - z(s_\nu)) \rangle + \langle z^{\nu+1} - z(s_\nu), C_{s_\nu}(z^\nu - z(s_\nu)) \rangle$$

or

$$0 \geq -\langle z^{\nu+1} - z(s_\nu), z^\nu - z(s_\nu) \rangle_{F_{s_\nu}} + \|z^{\nu+1} - z(s_\nu)\|_{F_{s_\nu}}^2. \tag{23}$$

By applying in (23) the Cauchy-Schwarz inequality with respect to $\langle \cdot, \cdot \rangle_{F_{s_\nu}}$ and $\|\cdot\|_{F_{s_\nu}}$ we get

$$\|z^{\nu+1} - z(s_\nu)\|_{F_{s_\nu}} \quad \leq \quad \|z^\nu - z(s_\nu)\|_{F_{s_\nu}}. \tag{24}$$

Now (17) follows from the equalities (see (19))

$$\langle x, y \rangle_{F_{s_\nu}} = (1 - s_\nu) \langle x, y \rangle_D, \quad \|x\|_{F_{s_\nu}} = \sqrt{1 - s_\nu} \|x\|_D. \tag{25}$$

For the equality (18), we first observe that from (23) and (25) we obtain

$$\langle z^{\nu+1} - z(s_\nu), z^\nu - z(s_\nu) \rangle_D \quad \geq \quad \|z^{\nu+1} - z(s_\nu)\|_D^2,$$

thus

$$\|z^{\nu+1} - z^\nu\|_D^2 = \|(z^{\nu+1} - z(s_\nu)) - (z^\nu - z(s_\nu))\|_D^2 =$$

$$\|z^{\nu+1} - z(s_\nu)\|_D^2 + \|z^\nu - z(s_\nu)\|_D^2 - 2\langle z^{\nu+1} - z(s_\nu), z^\nu - z(s_\nu) \rangle_D \leq$$

$$\|z^{\nu+1} - z(s_\nu)\|_D^2 + \|z^\nu - z(s_\nu)\|_D^2 - 2\|z^{\nu+1} - z(s_\nu)\|_D^2$$

from which (18) directly holds and the proof is complete. □

Unfortunately, we cannot continue to prove convergence of the sequence generated with Alg. 3.2 by following the same way as in Th. 5.6.1 in [2]. This is essentially related to the fact that in that proof the solution $\tilde{z}$ used is fixed with respect to the sequence index $\nu \geq 0$, whereas in our case the solution $z(s_\nu)$ is changing in each iteration.

But the above mentioned method can be considered in a more general context. For this, let D be an $n \times n$ diagonal positive definite matrix, i.e.

$$D = \mathrm{diag}(d_1, d_2, \ldots, d_n), \quad d_i > 0, \quad i = 1, \ldots, n \tag{26}$$

and z_{wmn} the unique minimal D-norm solution of the $\mathrm{LCP}(q, M)$ with q, M from (7).

Lemma 1. *If*

$$\hat{M} = D^{-\frac{1}{2}} M D^{-\frac{1}{2}}, \quad \hat{q} = D^{-\frac{1}{2}} q, \tag{27}$$

then the solutions of the problems $\mathrm{LCP}(q, M)$ *and* $\mathrm{LCP}(\hat{q}, \hat{M})$ *are connected through the following relations*

$$D^{\frac{1}{2}}\,\mathrm{SOL}(q, M) = \mathrm{SOL}(\hat{q}, \hat{M}) \quad \Leftrightarrow \quad D^{-\frac{1}{2}}\,\mathrm{SOL}(\hat{q}, \hat{M}) = \mathrm{SOL}(q, M). \tag{28}$$

Proof. Because M and $\hat{M}$ are SPD matrices and D is as in (26) we have the following sequence of equivalencies [2]

$$\mathrm{LCP}(q, M) \Leftrightarrow \min_{x \geq 0} \langle q, x \rangle + \frac{1}{2} \langle Mx, x \rangle \Leftrightarrow$$

$$\min_{x \geq 0} \left\langle D^{-\frac{1}{2}} q, D^{\frac{1}{2}} x \right\rangle + \frac{1}{2} \left\langle D^{-\frac{1}{2}} M D^{-\frac{1}{2}} (D^{\frac{1}{2}} x), (D^{\frac{1}{2}} x) \right\rangle \Leftrightarrow$$

$$\min_{y \geq 0} \langle \hat{q}, y \rangle + \frac{1}{2} \langle \hat{M} y, y \rangle, \quad (y = D^{\frac{1}{2}} x),$$

which completes the proof. □

Theorem 3.3. *Let the matrix* D *be as in* (26), $\hat{M}, \hat{q}$ *as in* (27) *and* $c > 0$ *a fixed number. We consider the Alg. 3.2 for the* $\mathrm{LCP}(\hat{q}, \hat{M})$, *with the splitting*

$$\hat{M} = \hat{B} + \hat{C}, \quad \hat{B} = \hat{M} + \frac{1}{c} I, \quad \hat{C} = -\frac{1}{c} I, \tag{29}$$

i.e.

$$(\hat{M} + \frac{1}{c} I) z^{\nu+1} + \hat{q} - (1 - s_\nu) \frac{1}{c} I z^\nu \geq 0 \perp z^{\nu+1} \geq 0. \tag{30}$$

Let z_{mn} *be the (unique) minimal (Euclidean) norm solution of* $\mathrm{LCP}(\hat{q}, \hat{M})$. *Then, the sequence* $(z^\nu)_{\nu \geq 0}$ *converges and*

$$\lim_{\nu \to \infty} z^\nu = z_{mn}, \quad \lim_{\nu \to \infty} D^{-\frac{1}{2}} z^\nu = z_{wmn}, \tag{31}$$

with z_{wmn} *defined in* (11).

Proof. Because the matrix $\hat{M}$ from (27) is SPSD, the $\mathrm{LCP}(\hat{q}, \hat{M})$ can be recasted as a quadratic program [2] of the form

$$\begin{aligned} &\text{minimize } \hat{q}^T x + \frac{1}{2} x^T \hat{M} x \\ &\text{subject to } x \geq 0. \end{aligned} \tag{32}$$

Rockafellar points out in [9] that convex programming problems such as quadratic programs can be put into the more general setting of finding zeros of maximal monotone operators. A maximal monotone operator producing a zero for an argument x if and only if $x \in \mathrm{SOL}(\hat{q}, \hat{M})$ is $F = \partial f$ i.e. the subgradient multifunction of

$$f(x) = \begin{cases} \hat{q}^T x + \frac{1}{2} x^T \hat{M} x & \text{if } x \geq 0 \\ \infty & \text{else.} \end{cases} \tag{33}$$

Thus the inclusion

$$0 \in F(x), \quad F = \partial f \tag{34}$$

is equivalent to LCP (7), where $F : H \to H$ is a maximal monotone operator (H is a Hilbert space). Rockafellar introduced the Proximal Point Algorithm (PPA) in [8] to solve such problems. It generates a sequence of iterates

$$z^{\nu+1} = J_{cF}(z^{\nu}), \quad \nu \geq 0, \tag{35}$$

where $c > 0$ and $J_{cF} = (I + cF)^{-1}$ is the resolvent of F. If $F^{-1}(0) \neq \emptyset$ it is known that the sequence $(z^{\nu})_{\nu\geq 0}$ converges weakly to *some* point in the set $V = F^{-1}(0)$ [8, 2].

In order to get strong convergence Xu considers a slight modification of (35) in [12]. This regularized PPA takes the form

$$z^{\nu+1} = J_{c_\nu F}(t_\nu u + (1 - t_\nu)z^{\nu} + e_\nu), \nu \geq 0, \tag{36}$$

where $t_\nu \in [0, 1]$, e_ν is the computational error, u is a fixed vector and $(c_\nu)_{\nu\geq 0}$ a sequence of positive numbers. Then, the following result is proved in [11].

Theorem 3.4. *([12], Th. 3.3)*
If the conditions
(C1) $\lim_{\nu\to\infty} t_\nu = 0, \quad \sum_{\nu=0}^{\infty} t_\nu = \infty$
(C3) $0 < \underline{c} \leq c_\nu \leq \bar{c}, \forall \nu \geq 0$
(C4') $\sum_{\nu=0}^{\infty} \lim_{\nu\to\infty} |1 - \frac{c_\nu}{c_{\nu+1}}| = 0$
(C5) $\sum_{\nu=0}^{\infty} \|e_\nu\| < \infty$
hold then, the sequence $(z^{\nu})_{\nu\geq 0}$ *generated by the Alg.* (36) *converges strongly and*

$$\lim_{\nu\to\infty} z^{\nu} = P_V(u), \tag{37}$$

where P_V *is the orthogonal projection onto the closed convex set* $V = F^{-1}(0)$*. For* $u = 0$ *the above limit* $P_V(u)$ *in* (37) *is the (unique) element from* V *with minimal Euclidean norm.*

In [2] the original PPA is applied to LCPs and in that context the iteration from Eq. (35) reads

$$(\hat{M} + \frac{1}{c}I)z^{\nu+1} + \hat{q} - \frac{1}{c}Iz^{\nu} \geq 0 \perp z^{\nu+1} \geq 0. \tag{38}$$

We observe that the Alg. (38) is Alg. 3.1 with the specific splitting $B = \hat{M} + \frac{1}{c}I,\ C = -\frac{1}{c}I$ and $\hat{M}, \hat{q}$ from (27). According to the result in Th. 3.1 the sequence $(z^{\nu})_{\nu\geq 0}$ converges to some element from $\mathrm{SOL}(\hat{q}, \hat{M})$.

We shall now consider the regularized PPA from Eq. (36) with $c_\nu = c$, $u = 0$, $e_\nu = 0$ and $t_\nu = s_\nu$. Then Eq. (36) becomes

$$z^{\nu+1} = J_{cF}((1 - s_\nu)z^{\nu}), \quad \nu \geq 0. \tag{39}$$

In comparison to (35) the argument of the resolvent is $(1 - s_\nu)z^{\nu}$ instead of z^{ν}. Substituting $(1 - s_\nu)z^{\nu}$ for z^{ν} in (38) leads to the regularized PPA for SPSD LCPs, which is exactly our Alg. (30). Note that $c_\nu = c > 0$ satisfies (C3) and (C4') and $e_\nu = 0$ satisfies (C5). Under the condition that s_ν satisfies (C1), it thus results from the above Th. 3.4 that the sequence generated by the regularized PPA for LCPs converges to z_{mn}, the minimum Euclidean norm solution of the $\mathrm{LCP}(\hat{q}, \hat{M})$, i.e.

$$\lim_{\nu\to\infty} z^{\nu} = z_{mn} \in \mathrm{SOL}(\hat{q}, \hat{M}). \tag{40}$$

If we now define

$$y^{\nu} = D^{-\frac{1}{2}} z^{\nu}, \quad \forall \nu \geq 0, \tag{41}$$

from (40) we obtain

$$\lim_{\nu\to\infty} y^{\nu} = D^{-\frac{1}{2}} z_{mn} = y^{*}. \tag{42}$$

From (40), (42) and (28) we obtain that $y^* \in \text{SOL}(q, M)$. Moreover, let $y \in \text{SOL}(q, M)$ be an arbitrary solution and (see again (28)) $z = D^{\frac{1}{2}} y \in \text{SOL}(\hat{q}, \hat{M})$. From the definition of z_{mn} and y^* we obtain

$$\|y^*\|_D = \|D^{-\frac{1}{2}} z_{mn}\|_D = \|z_{mn}\| \leq \|z\| = \|D^{-\frac{1}{2}} z\|_D = \|y\|_D,$$

i.e. $y^* = z_{wmn}$, the minimal D-norm solution of the problem (7). □

4 Numerical Tests

Numerical tests will be performed on an academic 2×2 example and a medium scale 256×256 example. Both examples are LCPs exhibiting SPSD system matrices with a non-empty nullspace. For each example the minimum Euclidean norm and the minimum D-norm for several different choices of D will be computed. Non-diagonal but SPD D will also be included to get numerical evidence on whether the theoretical results might be extendible to non-diagonal D. Each of these problems will be solved by the PPA, the regularized PPA and an implementation of the approach from Th. 3.2 termed "Tikhonov regularization" in the following. The implementation of the latter approach starts out with $s_0 = 1$, solves the LCP$(M + s_0 D, q)$ with the projected Gauss-Seidel method and uses the result as an initial solution for LCP$(M + s_{\nu+1} D, q)$, where $s_{\nu+1} = \frac{1}{2} s_\nu$. The regularized PPA is tested with the sequence $s_{\nu+1} = s_0 \frac{1}{\nu+1}$ satisfying (C1) but also with the sequence $s_{\nu+1} = \frac{1}{2} s_\nu$ violating (C1) on purpose. However, the first sequence in contrast to the other sequence starts from $s_0 = 10^{-8}$ in order to limit the number of iterations needed for s_ν to drop below machine precision.

4.1 Academic Example

The first test is constructed such that an analytic solution can be obtained. The system matrix is 2×2, positive semi-definite and symmetric with a non-empty nullspace:

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} x - \begin{pmatrix} 1 \\ 1 \end{pmatrix} \perp x \geq 0 \tag{43}$$

The set of solutions is

$$\text{SOL}(q, M) = \left\{ \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + (1 - \alpha) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \middle| \alpha \in [0; 1] \right\}. \tag{44}$$

The solution with minimum Euclidean norm is then obviously $(\frac{1}{2}, \frac{1}{2})^T$. A solution with minimum D-norm, where $D = \text{diag}(d_1, d_2)$ is diagonal can be easily determined to be $x^* = (\frac{d_2}{d_1+d_2}, \frac{d_1}{d_1+d_2})^T$. Tab. 1 shows the results of the PPA, the regularized PPA with two different sequences $(s_\nu)_{\nu \geq 0}$ and the Tikhonov regularization with $D = I$. The norm of the residuals in the first row indicate that all methods produced *some* solution of the LCP. The second row is the L2-norm of the error (difference to the analytic solution). The Tikhonov regularization cannot produce a solution with an error in the range of the machine precision whereas the PPA and regularized PPA can. The norm of the solutions of all three methods match the norm of the analytic solution x^*. When choosing D not to be a multiple of the identity but still diagonal, the results are qualitatively the very same as can be checked in Tab. 2. Here all methods, even the PPA and the regularized PPA both violating the condition (C1), seem to converge towards the minimum D-norm solutions.

4.2 Example from Multibody Contact Dynamics

The second example is a contact problem borrowed from rigid multibody dynamics with redundant constraints. The system matrix is 256×256, SPSD and rank-deficient with rank 242. Since we cannot compute an analytic

	Reg. reduction	PPA	Reg. PPA ($s_{\nu+1} = \frac{1}{2}s_\nu$)	Reg. PPA ($s_{\nu+1} = s_0\frac{1}{\nu+1}$)
$\|\min(Mx+q,0)\|_2$	$1.6078 \cdot 10^{-13}$	$2.8701 \cdot 10^{-13}$	$2.7163 \cdot 10^{-13}$	$8.9778 \cdot 10^{-13}$
$\|x - x^*\|_2$	$1.0451 \cdot 10^{-6}$	$9.6956 \cdot 10^{-12}$	$1.3977 \cdot 10^{-11}$	$4.2364 \cdot 10^{-11}$
$x^T x$	0.5	0.5	0.5	0.5

Table 1: Residuals, errors and norm of the solution of several iterative methods on Eq. (43) with $D = I$.

	Reg. reduction	PPA	Reg. PPA ($s_{\nu+1} = \frac{1}{2}s_\nu$)	Reg. PPA ($s_{\nu+1} = s_0\frac{1}{\nu+1}$)
$\|\min(Mx+q,0)\|_2$	$4.671 \cdot 10^{-13}$	$2.8576 \cdot 10^{-13}$	$6.9712 \cdot 10^{-13}$	$8.3513 \cdot 10^{-13}$
$\|x - x^*\|_2$	$1.0571 \cdot 10^{-6}$	$1.1326 \cdot 10^{-12}$	$2.2338 \cdot 10^{-12}$	$1.5643 \cdot 10^{-11}$
$x^T x$	0.36324	0.36324	0.36324	0.36324

Table 2: Residuals, errors and norm of the solution of several iterative methods on Eq. (43) with $D = \mathrm{diag}(0.64, 0.84)$.

solution we use the solution computed by the Tikhonov regularization as the reference solution x^*. Tab. 3 presents the results of all methods applied to the contact problem with $D = I$. Again all methods produce some solution and this solution is less than 10^{-7} away (in the L2-norm) from the reference solution obtained by the Tikhonov regularization. This rather high discrepancy can be explained by noticing the error that was present in the solution in the last section where the solution of the Tikhonov regularization also showed an error of this magnitude. Thus all methods seem to converge towards the minimum Euclidean norm solution even those variants that do not satisfy property (C1) like the PPA (where $s_\nu = 0$) and the regularized PPA where $s_{\nu+1} = \frac{1}{2}s_\nu$.

	Reg. reduction	PPA	Reg. PPA ($s_{\nu+1} = \frac{1}{2}s_\nu$)	Reg. PPA ($s_{\nu+1} = s_0\frac{1}{\nu+1}$)
$\|\min(Mx+q,0)\|_2$	$9.9610 \cdot 10^{-13}$	$9.8187 \cdot 10^{-13}$	$9.6741 \cdot 10^{-13}$	$9.9911 \cdot 10^{-13}$
$\|x - x^*\|_2$	-	$5.7336 \cdot 10^{-7}$	$5.7336 \cdot 10^{-7}$	$5.7336 \cdot 10^{-7}$
$x^T x$	0.057884	0.057884	0.057884	0.057884

Table 3: Residuals, errors and norms of the solutions of several iterative methods on a contact problem with $D = I$.

To demonstrate the applicability of Lemma 1 the next test uses a randomly generated positive, diagonal D denoted D_a and generated with the following Matlab commands:

```
RandStream.setDefaultStream(RandStream('mt19937ar','seed', 0));
D_a = diag(0.5+rand(size(M, 1), 1));
```

The precise D_a had a condition number of 2.9648. Tab. 4 lists the numerical results. Again the solution obtained by the Tikhonov regularization served as the reference solution x^*. All PPA variants seem to converge towards the same solution which is off by $\approx 7 \cdot 10^{-7}$ from the solution x^* of the Tikhonov regularization. The weighted norm of all computed solutions were the same within numerical precision.

When varying the weights we noticed that the PPA variants sometimes fail to produce the minimum norm solutions. This happened already when just slightly increasing the condition number of D. To demonstrate that, we consider another positive, diagonal D denoted D_b and generated with the following Matlab commands:

```
RandStream.setDefaultStream(RandStream('mt19937ar','seed', 0));
D_b = diag(0.5+rand(size(M, 1), 1)*10);
```

	Reg. reduction	PPA	Reg. PPA ($s_{\nu+1} = \frac{1}{2}s_\nu$)	Reg. PPA ($s_{\nu+1} = s_0\frac{1}{\nu+1}$)
$\|\min(Mx+q,0)\|_2$	$9.4206 \cdot 10^{-13}$	$1.0256 \cdot 10^{-12}$	$1.0158 \cdot 10^{-12}$	$1.0554 \cdot 10^{-12}$
$\|x - x^*\|_2$	-	$6.9199 \cdot 10^{-7}$	$6.9199 \cdot 10^{-7}$	$6.9199 \cdot 10^{-7}$
$x^T D_a x$	0.059958	0.059958	0.059958	0.059958

Table 4: Residuals, errors and norms of the solutions of several iterative methods on a contact problem with $D = D_a$.

The precise condition number of D_b was 19.1480. Tab. 5 lists the results. The solution of the Tikhonov regularization served as a reference solution x^* again. All solutions of the PPA variants have a slightly higher weighted norm than the solution of the Tikhonov regularization contradicting Th. 3.4. The specific conditions for the regularized PPA methods were steadily weakened recently [4, 12, 10, 11] but the main point is always the same as already noted in [4]: The sequence $(s_\nu)_{\nu\geq 0}$ must not tend to zero too fast in order to produce the same asymptotic behaviour as the Tikhonov regularization. Hence, we tried again with the sequence $s_{\nu+1} = s_0\frac{1}{\nu+1}$ but this time $s_0 = 1$. Obviously this sequence takes a long time for s_ν to be sufficiently small. Even though we did not complete this test the iterates after 1.5 million iterations had an error in the L2-norm of less than $2.2\cdot 10^{-5}$ and dropping which is already far less than the error 0.0017126 reported in Tab. 5. This might indicate that the condition on the error term $\sum_{\nu=0}^{\infty}\|e_\nu\| < \infty$ (C5) is too restrictive in practice. Research is going on to improve this condition [1]. However, in this publication the matrix changes in each iteration of the method.

	Reg. reduction	PPA	Reg. PPA ($s_{\nu+1} = \frac{1}{2}s_\nu$)	Reg. PPA ($s_{\nu+1} = s_0\frac{1}{\nu+1}$)
$\|\min(Mx+q,0)\|_2$	$9.5715 \cdot 10^{-13}$	$9.9997 \cdot 10^{-13}$	$9.9933 \cdot 10^{-13}$	$9.9996 \cdot 10^{-13}$
$\|x - x^*\|_2$	-	0.0017126	0.0017120	0.0017126
$x^T D_b x$	0.33478	0.33482	0.33482	0.33482

Table 5: Residuals, errors and norms of the solutions of several iterative methods on a contact problem with $D = D_b$.

Even though the theory for non-diagonal but SPD D is unclear we decided to test it. For the system matrix we chose one with the same eigenvalues as D_a. The dense matrix was generated by multiplying a random orthogonal matrix Q from the left and its transpose from the right:

```
RandStream.setDefaultStream(RandStream('mt19937ar','seed', 0));
[Q ~] = qr(rand(size(M, 1)));
D_c    = Q*D_a*Q';
```

The results are listed in Tab. 6. The weighted norms of all solutions are the same within numerical precision. However, the regularized PPA with the sequence $s_{\nu+1} = \frac{1}{2}s_\nu$ not satisfying (C1) does not seem to converge towards a solution. This can be explained by observing the back transformation in Eq. (41) may not be feasible and thus result in a high residual norm. Even this seems to affect only one of the PPA variants we expect the other ones to be also prone to that problem.

5 Conclusion

The proximal point algorithm can be readily applied to linear complementarity problems. The original PPA by Rockafellar [8] is not guaranteed to converge to a minimum norm solution. Several variants of the PPA method were proposed since. Lehdili and Moudafi proposed the Prox-Tikhonov in [4] with rather restrictive

	Reg. reduction	PPA	Reg. PPA ($s_{\nu+1} = \frac{1}{2}s_\nu$)	Reg. PPA ($s_{\nu+1} = s_0\frac{1}{\nu+1}$)
$\|\min(Mx+q,0)\|_2$	$9.9821 \cdot 10^{-13}$	$1.0079 \cdot 10^{-12}$	$1.0815 \cdot 10^{-5}$	$1.096 \cdot 10^{-12}$
$\|x - x^*\|_2$	-	0.00087361	0.00071069	0.00087361
$x^T D_c x$	0.066196	0.066196	0.066196	0.066196

Table 6: Residuals, errors and norms of the solutions of several iterative methods on a contact problem with $D = D_c$.

assumptions but strong convergence towards the minimum norm solution. Xu, Song et al and Wang weakened these conditions in [12, 10, 11]. However, the condition on the error is always $\sum_{\nu=0}^{\infty}\|e_\nu\| < \infty$. In [1] Boikanyo and Moroşanu considered the same algorithm under different conditions and could show strong convergence for weaker error conditions. In their approach in particular the sequence $(c_\nu)_{\nu\geq 0}$ approaches ∞ in the limit instead of being constant as in our tests. As our tests indicate (C5) seems to be too restrictive in practice. Also the approach per se seems to require a slow convergence of the sequence $(s_\nu)_{\nu\geq 0}$ in order to guarantee convergence towards the minimum norm solution. Considering for example the sequence $s_\nu = \frac{1}{\nu}$ with $s_0 = 1$ the test needed $1.5 \cdot 10^6$ iterations to reduce the error to $2.2 \cdot 10^{-5}$. This makes the method tedious to use. A theoretical result for an error bound when violating (C1) would be interesting. This would enable us to select a faster converging sequence and still obtain a result with a controlled error. As it currently stands the method is not reliably usable.

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