Numerical Simulation of Coupled Electro-Mechanical systems using the Finite Element Method

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Outline

- Motivation
  - Micromachined devices
  - Modeling requirements

- Theory
  - Electrostatic and mechanical field
  - Force calculation – Virtual Work Principle
  - Iterative coupling scheme
  - Moving mesh technique

- Application
  - Micromachined mirror
  - Ultrasound transducer (CMUT)

- Conclusion
Micromachined mirror for optical equipment (video projector, network router)

Structure is produced directly on silicon chips

Very fast switching cycles

Typical dimensions: 200 $\mu$m x 120 $\mu$m x 2 $\mu$m

$\Rightarrow$ Very large aspect ratio (1:100)

$\Rightarrow$ Deformation often larger than plate-thickness
Motivation - CMUT

- Capacitive Micromachined Ultrasonic Transducers (CMUT)
- Electrically excited membranes generate emerging ultrasonic wave
- Used for medical diagnostics
- Very thin membranes: 0.5 – 1 µm
- Advantage: Transducer cell and electronics are on one single chip
Governing Equations

Electrostatic Field

\[ \nabla \cdot (\epsilon \nabla V_e) = -q_e \]

- FE-Matrix formulation:

\[ K_{Ve}(u) V_e = f_q \]

- Note: Electric field computation has to be performed on updated geometry (Updated Lagrangian formulation)

Mechanical Field

\[ \mathcal{B}^T[c] \mathcal{B} u + f_V = \rho \ddot{u} \]

\[ M_u \dddot{u} + K_u u = f_u \]

- \( u \): mechanical displacement
- \( \dddot{u} \): mechanical acceleration
- \( f_V \): mechanical volume force
- \( [c] \): tensor of mechanical moduli
- \( \mathcal{B} \): differential operator
- \( \rho \): density

- \( M_u \): mechanical mass matrix
- \( K_u \): mechanical stiffness matrix
- \( u \): nodal displacement vector
- \( \dddot{u} \): nodal acceleration vector
Electrostatic Force Calculation

- Assume rigid body with $\varepsilon_r >> \varepsilon_0$ in electric field
- Take element layer around body with fixed nodes

Goal: Calculation of force $F_i$ at node $i$

Known: Nodal electric potential $V_e$

Idea: Force is derivative of work w.r.t. direction $r$

*(Virtual Work Principle)*

\[ F_r = \frac{\partial W}{\partial r} \]

$F_r$ : force in direction of $r$

$W$ : work

$r$ : direction of force
Electrostatic Force Calculation (cont.)

- Derivative of electric work is

\[ F_r = \frac{\partial}{\partial r} \int_{\Omega} \frac{\epsilon_0}{2} \mathbf{E} \cdot \mathbf{E} \ d\Omega \]

- Sum up forces of all elements $n_e^i$ neighboring node $i$

- Resulting force on node $i$ in direction $r$

\[ F_r^i = \sum_{n=1}^{n_e^i} \left( \int_{\Omega_i} \epsilon_0 \mathbf{E} \frac{\partial \mathbf{E}}{\partial r} \ d\Omega + \int_{\Omega_i} \frac{\epsilon_0}{2} \mathbf{E} \cdot \mathbf{E} \ \frac{\partial (d\Omega)}{\partial r} \right) \]

Diagram: Moving node, fixed nodes, element forces, elements neighboring node $i$.
Coupling Scheme

- Resulting system with Right-Hand-Side coupling

\[
\begin{pmatrix}
K_u & 0 & 0 \\
0 & K_{Ve}(u^i) & 0 \\
0 & 0 & K_S
\end{pmatrix}
\begin{pmatrix}
u_{i+1} \\
V_{ei+1} \\
u_{S_{i+1}}
\end{pmatrix}
+ \begin{pmatrix}
M_u & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\dot{u}_{i+1} \\
\dot{V}_{ei+1} \\
\dot{u}_{S_{i+1}}
\end{pmatrix}
= \begin{pmatrix}
f^i_{u+1}(V_{ei}) \\
f^i_{q+1} \\
0
\end{pmatrix}
\]

- No off-diagonal entries in matrices due to RHS-coupling
- Only \( K_{Ve} \) has to be recalculated each time (geometric non-linearity) and the coupling force \( f^i_{u+1}(V_{ei}) \)
- Stopping criterion is based on relative change of mechanical displacements
  \[
  \frac{\|u_{i+1}^i-u_i^i\|_2}{\|u_{i+1}^i\|_2} < \delta
  \]
- Coupling scheme and force calculation were implemented in FE code of department (bachelor thesis)
Coupling Scheme (cont.)

- All individual fields are iteratively solved in a sequential order
- Coupling is done only via Right-Hand-Side terms and boundary conditions
Problem setup:

1.) Electric field:

2.) Mechanical field:

3.) Grid smoothing:
Moving Mesh Technique

- Due to large deformations, intersection of elements can occur

- Idea: Solve additional equation with interface-deformations as Dirichlet boundary condition (mesh smoothing)
1.) Laplace Smoother

- Solve Laplace equation to distribute deformations uniformly over air region

\[
\Delta u_x = \frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} = 0
\]

\[
\Delta u_y = \frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2} = 0
\]

- Properties
  - Fast and simple (only few solver iterations needed)
  - Max. deformation of only 30% of initial gap length, as two directions are solved independently
2.) Modified Mechanical Problem

- Solve mechanical problem in distorted domain
  \[ B^T [\tilde{c}] B \mathbf{u} = 0 \]

- a) Adapt mechanical stiffness relative to Jacobian determinant
  \[ \tilde{c} = \frac{c}{|J|} \sim \text{element size} \]

- b) Adapt mechanical stiffness relative to mean element strain
  \[ \tilde{c} = \frac{c}{\varepsilon^2} \left( \varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2 \right) \frac{4}{4} \]

- Advantage: Deformations up to 70% of initial gap length possible
Application – Micro Mirror

- Magnified model:
  - Mirror: 9 mm x 3 mm x 40 µm
  - Air gap: 200 µm

- Different cross-section dimensions of the clamping-spring are compared

![Graph showing displacement vs. force with different simulation curves and a legend indicating material parameters: E = 1.62e11 N/m, µ = 0.001. The graph includes measurements and six simulation runs (sim.1 to sim.6).]
Application – Micro Mirror (cont.)

- Jump response to a single-pulse signal
- Only upper electrodes are visualized
Application – CMUT

- Additional coupling to acoustic field required
- Simultaneous excitation of all 5 membranes
Conclusion

- Iterative coupling scheme allows simple extension of existing single-field simulations towards multi-field applications
- Simple calculation of nodal electrostatic forces by the Virtual Work Principle (also usable for magnetic field simulation)
- Two different approaches for grid smoothing
- Ongoing work
  - Nonlinear mechanics for large deformations
  - Additional mechanical-acoustic coupling
  - Electro-magneto-mechanical coupling in power transformers
    (Master Thesis)

Acknowledgement

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References


Electrostatic Field

- Physical equations
  \[ \nabla \cdot D = q_e \]
  \[ D = \varepsilon E \]

- Potential formulation
  \[ E = -\nabla V_e \]
  \[ \nabla \cdot (\varepsilon \nabla V_e) = -q_e \]

- FE-Matrix formulation

\[
\begin{align*}
K_{Ve}(u) V_e &= f_q \\
K_{i,j} &= \int_{\Omega} \varepsilon \nabla N_i(x) \cdot \nabla N_j(x) d\Omega \\
f_{j} &= \int_{\Omega} -q_e N_i(x) d\Omega 
\end{align*}
\]

- Note: Field computation has to be performed on updated geometry (Updated Lagrangian formulation)

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>(D)</td>
<td>electric displacement</td>
</tr>
<tr>
<td>(E)</td>
<td>electric field intensity</td>
</tr>
<tr>
<td>(V_e)</td>
<td>scalar electric potential</td>
</tr>
<tr>
<td>(q_e)</td>
<td>electric volume charge</td>
</tr>
<tr>
<td>(\varepsilon)</td>
<td>electric permittivity</td>
</tr>
<tr>
<td>(K_V)</td>
<td>electric stiffness matrix</td>
</tr>
<tr>
<td>(V_e)</td>
<td>nodal potential vector</td>
</tr>
<tr>
<td>(N_j)</td>
<td>nodal ansatz function</td>
</tr>
</tbody>
</table>
Mechanical Field

- Navier's equation
  \[ \mathcal{B}^T \sigma + f_V = \rho \ddot{u} \]

- Stress-strain-relation
  \[ \sigma = [c] S \]
  \[ S = \mathcal{B} u \]

- Resulting equation
  \[ \mathcal{B}^T [c] \mathcal{B} u + f_V = \rho \ddot{u} \]

- FE-Matrix formulation

\[
\begin{align*}
\mathbf{M}_u \dddot{u} + \mathbf{K}_u u &= \mathbf{f}_u \\
\mathbf{M}_{i,j} &= \int_{\Omega} \rho \mathbf{N}_i(x) \cdot \mathbf{N}_j(x) d\Omega \\
\mathbf{K}_{i,j} &= \int_{\Omega} (\mathcal{B}_i^T) [c] (\mathcal{B}_j) d\Omega \\
\mathbf{f}_j &= \int_{\Omega} \mathbf{N}_j^T f_V d\Omega
\end{align*}
\]
Appendix A – FEM Overview

- Scheme of the Finite Element Approximation:

  - Step 1: Electrostatic PDE
  \[ \nabla \cdot (\varepsilon \nabla V_e) = -q_e \]
  - Step 2: Multiply with test function \( w \)
  \[ \int_{\Omega} w \nabla \cdot (\varepsilon \nabla V_e) d\Omega = -\int_{\Omega} q w d\Omega \]
Appendix A – FEM Overview

- Step 3: Apply integral theorem of Green

\[ \int_{\Omega} V_1 \Delta V_2 d\Omega + \int_{\Omega} \nabla V_1 \cdot \nabla V_2 d\Omega = \oint_{\Gamma} V_1 \frac{\partial V_2}{\partial n} d\Gamma \]

\[ \int_{\Omega} \epsilon \nabla w \cdot \nabla V_e d\Omega - \oint_{\Gamma} w \frac{\partial V_e}{\partial n} d\Gamma = \int_{\Omega} qwd\Omega \]

- Step 4: By choosing \( w=0 \) on the boundary, we arrive at the continuous weak form of the PDE

\[ \int_{\Omega} \epsilon \nabla w \cdot \nabla V_e d\Omega = \int_{\Omega} qwd\Omega \]
Appendix A – FEM Overview

- Step 5: Approximate test function by nodal-ansatz functions $N_i$ and according coefficients $V_{ei}$ on all interior nodes $n_{int}$

$$V_e(x) \approx \sum_{i=0}^{n_{int}} V_{ei}N_i(x) + \sum_{i=n_{int}+1}^{n_n} V_{ei}N_i(x)$$

$$w(x) \approx \sum_{i=0}^{n_{int}} \phi_i(x)$$

- Step 6: Discrete weak form of the PDE

$$\int_{\Omega} \epsilon \nabla \left( \sum_{i=0}^{n_{int}} N_i(x) \right) \cdot \nabla \left( \sum_{j=0}^{n_{int}} V_{ei}N_j(x) \right) d\Omega = \int_{\Omega} q \left( \sum_{i=0}^{n_{int}} N_i(x) \right) d\Omega$$
Appendix A – FEM Overview

- Step 7: Reorder terms and reformulate as linear algebraic system of equations

\[
\sum_{i=0}^{n_{int}} \sum_{j=0}^{n_{int}} \epsilon V_{e_i} \int_{\Omega} (\nabla N_i(x)) \cdot (\nabla N_j(x)) \, d\Omega = \int_{\Omega} q \left( \sum_{i=0}^{n_{int}} N_i(x) \right) \, d\Omega
\]

\[
K V_e = f
\]

\[
K_{i,j} = \int_{\Omega} \epsilon \nabla N_i(x) \cdot \nabla N_j(x) \, d\Omega
\]

\[
f_{j} = \int_{\Omega} q_e N_i(x) \, d\Omega
\]
Appendix C – Force Calculation

\[ F_r^i = \sum_{n=1}^{n_e^i} \left( \int_{\Omega_i} \frac{\varepsilon_0}{2} \mathbf{E} \cdot \mathbf{E} \frac{\partial |\mathbf{J}|}{\partial r} \, d\xi d\eta d\zeta - \int_{\Omega_i} \varepsilon_0 \mathbf{E} (\mathbf{J}^{-1})^T \frac{\partial (\mathbf{J}^T)}{\partial r} \mathbf{E} |\mathbf{J}| \, d\xi d\eta d\zeta \right) \]

Variation of electric field

variation of element size