Towards Intuitionistic Dynamic Logic

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A Kripke model \( \mathcal{K} \) (KM) is a pair \( \mathcal{K} = (W, m) \), where

- \( W \) a nonempty set of states
- \( m : \) is a function with
  1. for every atomic propositional \( a \): \( m(a) \subseteq W \)
  2. for every atomic program \( p \): \( m(p) \subseteq W \times W \)
intuitionistic Kripke models

An intuitionistic Kripke model \( \mathcal{R} \) (iKM) is a tripel \( \mathcal{R} = (\mathcal{W}, \preceq, m^\mathcal{R}) \), where

- \( \mathcal{W} \) a nonempty set of states
- \( \preceq \subseteq \mathcal{W} \times \mathcal{W} \) the intuitionistic (reflexive and transitive) accessibility relation
- \( m^\mathcal{R} \) : is a function with
  1. for every atomic propositional \( a \): \( m^\mathcal{R}(a) \subseteq \mathcal{W} \) monotonic in relation to \( \preceq \),
    this means: \( w \in m^\mathcal{R}(a) \) and \( w \preceq w' \) implies \( w' \in m^\mathcal{R}(a) \).
  2. for every atomic program \( p \): \( m^\mathcal{R}(p) \subseteq \mathcal{W} \times \mathcal{W} \)
Extension of the denotation function

- \( m^\mathcal{R}(\pi \cup \rho) := m^\mathcal{R}(\pi) \cup m^\mathcal{R}(\rho) \)
- \( m^\mathcal{R}(\pi; \rho) := m^\mathcal{R}(\pi) \circ m^\mathcal{R}(\rho) \) (composition of relations)
- \( m^\mathcal{R}(\pi^*) := \bigcup_{i \in \mathbb{N}} m^\mathcal{R}(\pi)^i \) (reflexive-transitive closure)
- \( m^\mathcal{R}(\varphi) := \{ (w, w) \mid w \in m^\mathcal{R}(\varphi) \} \)
- \( m^\mathcal{R}(\bot) := \emptyset \)
- \( m^\mathcal{R}(\psi \land \eta) := m^\mathcal{R}(\psi) \cap m^\mathcal{R}(\eta) \)
- \( m^\mathcal{R}(\psi \lor \eta) := m^\mathcal{R}(\psi) \cup m^\mathcal{R}(\eta) \)
- \( m^\mathcal{R}(\llbracket \pi \rrbracket \psi) := \{ w \in W^\mathcal{R} \mid \{ w' \mid (w, w') \in m^\mathcal{R}(\pi) \} \subseteq m^\mathcal{R}(\psi) \} \)
- \( m^\mathcal{R}(\llangle \pi \rrangle \psi) := \{ w \in W^\mathcal{R} \mid \{ w' \mid (w, w') \in m^\mathcal{R}(\pi) \} \cap m^\mathcal{R}(\psi) \neq \emptyset \} \)
- \( m^\mathcal{R}(\psi \rightarrow \eta) := \{ w \in W^\mathcal{R} \mid \{ w' \mid w \preceq^\mathcal{R} w' \} \cap m^\mathcal{R}(\psi) \subseteq m^\mathcal{R}(\eta) \} \)
Separation of Necessitas and Possibilitas

As well known in classical PDL \([\pi]\) and \(\langle\pi\rangle\) can be defined from each other:

- \(\Box \pi \varphi \leftrightarrow \neg \langle\pi\rangle
\neg \varphi\)
- \(\Diamond \pi \varphi \leftrightarrow \neg [\pi] \neg \varphi\)
Counterexample

At state $w_1$ the instances

$\vdash [p]a \rightarrow \neg \langle p \rangle \neg a$

$\vdash \langle p \rangle a \rightarrow \neg [p] \neg a$

are refuted.
Counterexample

At state $w_1$ the directions

1. $\neg \langle \langle \langle \langle \langle p \rangle \rangle \rangle \rangle \rangle \neg a \rightarrow \langle \langle \langle \langle \langle p \rangle \rangle \rangle \rangle \rangle a$

2. $\neg \langle \langle p \rangle \rangle \neg a \rightarrow \langle \langle p \rangle \rangle a$

are refuted.

Diagram:

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W1  p  W2  ⬅️  W3
∅   ⬕️  ∅   {a}
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Induction principles

The Segerberg induction schema consists of the following two axioms

\[ \varphi \land [\pi^*](\varphi \rightarrow [\pi] \varphi) \rightarrow [\pi^*] \varphi \] (Necessitas form)

\[ \langle \pi^* \rangle \varphi \rightarrow \varphi \lor \langle \pi^* \rangle (\neg \varphi \land \langle \pi \rangle \varphi) \] (Possibilitas form)

Theorem

*The Necessitas form is valid in iPDL, the Possibilitas form is not.*
Proof

- At Necessitas form: induction on $k$ for
  $$m^R(\varphi \land \Box^* \Box (\varphi \rightarrow \Box \Box \varphi)) \subseteq m^R(\Box^{k} \Box \varphi)$$

- At Possibilitas form: The instance
  $$\langle p^* \rangle a \rightarrow a \lor \langle p^* \rangle ((a \rightarrow \bot) \land \langle p \rangle a)$$
  is refuted in $w_1$:
Definability

Lemma

In every intuitionistic Kripke model \( \mathcal{R} = (W^\mathcal{R}, \preceq^\mathcal{R}, m^\mathcal{R}) \) holds

1. \( m^\mathcal{R}(\llbracket \psi \rrbracket \bot) = W^\mathcal{R} \setminus m^\mathcal{R}(\psi) \)
2. \( m^\mathcal{R}(\llbracket \pi \rrbracket \varphi) = m^\mathcal{R}(\llbracket (\llbracket \pi \rrbracket \llbracket \varphi \rrbracket \bot) \rrbracket \bot) \)
3. \( m^\mathcal{R}(\psi \land \eta) = m^\mathcal{R}(\llbracket \psi \rrbracket \eta) = m^\mathcal{R}(\llbracket (\llbracket \psi \rrbracket \llbracket \eta \rrbracket \bot) \rrbracket \bot) \)
4. \( m^\mathcal{R}(\psi \lor \eta) = m^\mathcal{R}(\llbracket (\llbracket \psi \rrbracket \bot) \rrbracket \eta) \)
5. \( m^\mathcal{R}(\llbracket \pi \cup \rho \rrbracket \varphi) = m^\mathcal{R}(\llbracket \pi \rrbracket \varphi \land \llbracket \rho \rrbracket \varphi) \)
6. \( m^\mathcal{R}(\llbracket \pi \cup \rho \rrbracket \varphi) = m^\mathcal{R}(\llbracket \pi \rrbracket \varphi \lor \llbracket \rho \rrbracket \varphi) \)
7. \( m^\mathcal{R}(\llbracket \pi \rrbracket \llbracket \rho \rrbracket \varphi) = m^\mathcal{R}(\llbracket \pi ; \rho \rrbracket \varphi) \)
8. \( m^\mathcal{R}(\llbracket \pi \rrbracket \llbracket \rho \rrbracket \varphi) = m^\mathcal{R}(\llbracket \pi ; \rho \rrbracket \varphi) \)
Towards Intuitionistic Dynamic Logic

Definability

Consequences

- it is sufficient to assume that a formula $\varphi$ contains only \( \bot, \rightarrow, *, ;, \cup, [ ] \) and ? as logical connectors.
- The formula $a \lor [a?] \bot$ is valid, which is very counterintuitive.

$\Rightarrow$ No significant disjunctive or existential property possible.
Small model property

Theorem
Every formula $\varphi$, which is satisfiable in a iPDL-model $\mathcal{K}$, is also satisfiable in a model with less than $2^c \cdot |\varphi|$ states, for some constant $c$.

- short proof by simulating the behaviour of iPDL formulae in PDL
- Open problem: How to construct the Fisher-Ladner closures and formulate of a filtration lemma for iPDL?
- Validity problem is decidable.
Proof

- choose an atomic program $p$ not included in $\varphi$.
- do following recursive translation of $\varphi$ to $\varphi^T$:
  - For atomic propositions $a$ [atomic programs $q$]:
    \[ a^T = \llbracket p^* \rrbracket a \quad [q^T = q] \]
  - $(\varphi \rightarrow \psi)^T = \llbracket p^* \rrbracket (\varphi^T \rightarrow \psi^T)$
  - move the $T$ inward for the other connectors
    (example $(\eta?)^T = (\eta^T)?$)
- Define a PDL-model $\mathcal{R}^T = (W^T_{\mathcal{R}}, m^T_{\mathcal{R}})$ by
  - $W^T_{\mathcal{R}} := W^\mathcal{R}$
  - Let $a$ be a propositional. Set $m^T_{\mathcal{R}}(a) := m^\mathcal{R}(a)$
  - Let $q$ be an atomic program with $p \neq q$.
    Set $m^T_{\mathcal{R}}(q) := m^\mathcal{R}(q)$.
  - Set $m^T_{\mathcal{R}}(p) := \llbracket \leq \rrbracket^\mathcal{R}$.
Proof (cont’d)

- Lemma: \( w \in m^\phi(\varphi) \) if and only if \( w \in m^\varphi_T(\varphi^T) \)

- By the small model theorem of the PDL one can choose a (PDL-) model \( \mathcal{L} = (\mathcal{W}, m_\mathcal{L}) \), which satisfies:
  - There is a state \( u \in \mathcal{W} \), such that \( u \in m_\mathcal{L}(\varphi^T) \)
  - \( |\mathcal{W}| \leq 2|\varphi^T| \leq 2^c|\varphi| \)

- Transform \( \mathcal{L} = (\mathcal{W}, m_\mathcal{L}) \) back to an iPDL-model by defining
  - \( \mathcal{W}^{\xi,P} := \mathcal{W}_\xi \)
  - For each atomic propositional \( a \): \( m^{\xi,P}_a := m_\xi([p^*]a) \)
  - For each atomic program \( q \): \( m^{\xi,P}(q) := m_\xi(q) \)
  - \( \preceq^{\xi,P} := (m_\xi(p))^* \)

- Lemma: \( w \in m^{\xi,P}(\varphi) \) if and only if \( w \in m_\xi(\varphi^T) \)
Monotonicity

A formula $\varphi$ is called monotonic if in every iPDL-model $\mathcal{R}$ and any of its worlds $u, v \in W^\mathcal{R}$: $u \leq^\mathcal{R} v$, $u \in m^\mathcal{R}(\varphi)$ implies $v \in m^\mathcal{R}(\varphi)$.

Counterexample: Consider the formula $\langle \langle \langle \langle \langle p \rangle \rangle \rangle \rangle a$ at states $w_1$ and $w_3$
Consequences of nonmonotonic formulae

- Failure of the $\Rightarrow\Rightarrow$ rule in Gentzen style systems. Example: $\langle p \rangle \top, \top \Rightarrow \langle p \rangle \top$ is valid, while $\langle p \rangle \top \Rightarrow \top \rightarrow \langle p \rangle \top$ is not.

- Failure of substitution: The valid formula from iPDL are not closed under substitution of atomic propositions for arbitrary formulae.
  $(a \rightarrow (b \rightarrow a)$ is valid, while $\langle p \rangle \top \rightarrow (\top \rightarrow \langle p \rangle \top$ is not)
iterated property: If $\langle \pi^* \rangle \psi$ is valid, then there is a natural number $k$ such that $\langle \pi^k \rangle \psi$ is also valid.

disjunctive property: If $\varphi \lor \psi$ is valid, then one of $\varphi$ or $\psi$ is valid.

The disjunctive property fails for iPDL: Consider the formula $\langle p \rangle \top \lor \lbrack p \rbrack \bot$ (both parts of the formula are nonmonotonic).

extended version: Restriction to subclasses of iPDL-models given by sets of (Harrop-) formulae ($F \models \langle \pi^* \rangle \psi$ then there is a $k \in \mathbb{N}$ s.t. $F \models \langle \pi^k \rangle \psi$)

no nontrivial form of the both versions.
Harrop formulae

Definition
The set $\mathbb{H}$ of Harrop formulae is recursive defined as follows:

- every atomic proposition is in $\mathbb{H}$
- if $\psi, \eta \in \mathbb{H}$ then is also $\psi \land \eta \in \mathbb{H}$
- if $p$ is an atomic program and $\varphi \in \mathbb{H}$ then is also $\left[p\right] \varphi \in \mathbb{H}$
- if $\left[\pi\right] \varphi, \left[\rho\right] \varphi \in \mathbb{H}$ then is also $\left[\pi \cup \rho\right] \varphi \in \mathbb{H}$
- if $\left[\pi\right]\left[\rho\right] \varphi \in \mathbb{H}$ then is also $\left[\pi; \rho\right] \varphi \in \mathbb{H}$
- if $\varphi \in \mathbb{H}$ and $\psi$ is a monotonic formula, then is also $\psi \rightarrow \varphi \in \mathbb{H}$
The definition of $\mathbb{H}$ is very strong with respect to the iterative property:

**Lemma**

Let $F \subseteq \mathbb{H}$ be a set of Harrop formulae, $\pi$ a program and $\eta$ a monotonic formula, then holds: if $F \models \langle \langle \langle \langle \langle \langle \pi^* \rangle \rangle \rangle \rangle \rangle \eta$, then $F \models \eta$.

Also a disjunctive property can be proven:

**Lemma**

Let $F \subseteq \mathbb{H}$ be a set of Harrop formulae and $\eta$ and $\psi$ monotonic formulae.

If $F \models \eta \lor \psi$, then we have $F \models \eta$ or $F \models \psi$. 
Open problems

- Can the definition of $\mathbb{H}$ be extended to get nontrivial results?
- When is the monotonicity condition not required?
- Is there a valid formula $\langle \pi^* \rangle \psi$ such that $\psi$ is not monotonic?
Definition
The modal translation $\overline{\varphi}$ of a formula $\varphi$, resp. $\overline{\pi}$ of a program $\pi$, is recursively defined, as follows: Let $\varphi, \eta, \psi$ be formulae and $\pi, \rho, \sigma$ be programs

- for atomic propositions $a$: $\overline{a} := a$ ($\overline{\bot} := \bot$)
- for atomic programs $p$: $\overline{p} := p$
- let $\varphi = \psi \rightarrow \eta$: then $\overline{\varphi} := \overline{\psi} ? \overline{\eta}$
- let $\varphi = \psi \circ \eta$: then $\overline{\varphi} := \overline{\psi} \circ \overline{\eta}$ ($\circ$ out of $\lor$, $\land$)
- let $\pi = \rho \circ \sigma$: then $\overline{\pi} := \overline{\rho} \circ \overline{\sigma}$ ($\circ$ out of $;$, $\cup$)
- let $\pi = \rho^*$: then $\overline{\pi} := (\overline{\rho})^*$
- let $\pi = \eta?$: then $\overline{\pi} := \overline{\eta}$?
- let $\varphi = [\rho] \eta$: then $\overline{\varphi} := [\overline{\rho}] \overline{\eta}$
- let $\varphi = \langle \rho \rangle \eta$: then $\overline{\varphi} := \langle \overline{\rho} \rangle \overline{\eta}$
Properties

- conceptual impure: modalities can be introduced in nonmodal formulae
- unusual since the $\rightarrow$ is replaced while everything else is untouched.

Theorem

Let $\varphi$ be a formula. Then $\varphi$ is valid in PDL iff $\overline{\varphi}$ is valid in iPDL.
so far no useful translation for the test operator.
  ▶ different behaviour when occurring in [ ] or ⟨ ⟩
  ▶ undeterminable behaviour in complex programs

The tried out negative translations preserve "only" classical tautologoids

The translated Segerberg axioms have mostly instances which are not valid.
Example (test free)

Consider the following negative translation of usual brand:

- For atomic propositions $a$: $a' := \neg\neg a$ ($\bot' = \bot$)
- For atomic programs $p$: $p' := p$
- $(\varphi \land \psi)' = \varphi' \land \psi'$
- $(\varphi \lor \psi)' = \neg(\neg\varphi' \land \neg\psi')$
- $(\varphi \rightarrow \psi)' = \varphi' \rightarrow \psi'$
- $(\pi; \rho)' = \pi'; \rho'$
- $(\pi \cup \rho)' = \pi' \cup \rho'$
- $(\pi^*)' = (\pi')^*$
- $([\pi] \varphi)' = \neg\neg[\pi'] \varphi'$
- $(\langle \pi \rangle \varphi)' = \neg\neg[\pi'] \neg\varphi'$
Example (cont’d)

- Segerberg axiom 6: $[\pi; \rho] \varphi \leftrightarrow [\pi] [\rho] \varphi$ (valid in iPDL!)
- Translation: $\neg\neg[\pi; \rho] \neg\neg\varphi \leftrightarrow \neg\neg[\pi] \neg\neg[\rho] \neg\neg\varphi$
- The instance $\neg\neg[\pi; \rho] \neg\neg\varphi \leftrightarrow \neg\neg[\pi] \neg\neg[\rho] \neg\neg\varphi$ is not valid:

$$
\begin{array}{cccc}
W_1 & \xrightarrow{p} & W_2 & \xrightarrow{q} W_3 \\
W_4 & \xrightarrow{q} & W_5 \\
\end{array}
$$
Thanks, for your attention.