Numerical Analysis of Stokes Equations with Improved LBB Dependency

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common testing domains for Simulations are **spheres, cubes**

But what are **appropriate** testing domains for CFD ???

**Flow in tubes / pipelines**
Water flow in nature

Typical Domains in Computational Fluid Dynamics
Domain-Dependency of Stokes Problem

Motivation
Improved A-priori Results
LBB-friendly Methods
Summary and Acknowledgments
Weather Simulation

weather is determined in troposphere (8 – 15 km)
Medical Application

e.g. simulation of aneurysms at Univ. Erlangen (J. Goetz)
Domains with large aspect ratio

Definition (Aspect Ratio $a$)

\[
a \approx \frac{\text{largest elongation}}{\text{smallest elongation}}
\]

- $a \propto$ length of tube, pipeline
- $a \propto 10^2$ for weather simulation above Germany
- $a \propto 10^4$ for simulation of whole Baltic Sea
- $a \propto 10^1 - 10^2$ in medical applications
- $a = 1$ for cubes, spheres

⇒ Even if the simulation restricts to a smaller subdomain, dependency on aspect ratio should be avoided!

⇒ Do tests on Stretched Domains $\Omega_a = \Omega \times \{a\}$ with $a(\Omega) \approx 1$
We consider compressible Stokes Equations

\[(Du, Dv) - (\text{div } v, p) = (f, v) \quad \forall v \in X\]
\[(\text{div } u, q) = (g, q) \quad \forall q \in Y\]

with \(f \in L^2(\Omega), g \in H^1_0(\Omega)\)

- \(X = H^1_0(\Omega), \quad Y = L^2(\Omega)_0 = \{ q \in L^2(\Omega) : \int_\Omega q \, dx = 0 \}\)
- \(\|u\|_m^2 = \sum_{\alpha \leq m} \|D^\alpha u\|_{L^2(\Omega)}^2, \quad |u|_m^2 = \sum_{\alpha = m} \|D^\alpha u\|_{L^2(\Omega)}^2\)
- unique solution guaranteed by LBB (or inf-sup) condition:

\[\exists L > 0 \quad \forall p \in Y : \quad L \|p\| \leq \sup_{v \in X} \frac{(\text{div } v, p)}{\|Dv\|} = \|Dp\|_{-1} \leq \|p\|\]
Standard Estimates with LBB Dependency

Regularity

\[ |u|_1 \leq \frac{c}{L} \|g\| \quad \text{(Bogovskij)} \]

Error bound for conforming FEM

\[ \|u - u_h\|_1 + \|p - p_h\| \leq c \left( c_T, \frac{1}{L} \right) (\|f\| + \|g\|_1) \]

- \( c_T \) is given by regularity of the discretization
- \( \Rightarrow \) domain dependency of solution and approximation \( \propto \frac{1}{L} \)
Domain Dependency of LBB Constant $L$

- **good:** diam $\Omega = R$ and $\Omega$ star-shaped w.r.t radius $r$

\[ 1 \geq L \geq c \left( \frac{r}{R} \right)^{n+1} \quad \text{(Bogovskij, 1979)} \]

- **bad:** Stretched Domains $\Omega_a = \Omega \times \{a\}$

\[ L(\Omega) \leq \frac{c(\omega)}{a} \quad \text{(Dobrowolski, 2005)} \]

- **bad:** $\Omega_a = \{ y \in \mathbb{R}^n : \left( \frac{y_1}{\alpha_1}, \ldots, \frac{y_n}{\alpha_n} \right) \in \Omega \}$, max $\alpha_i = a$

\[ \frac{L(\Omega)}{a} \leq L(\Omega_a) \leq \frac{c(\Omega)}{a} \quad \text{(Dobrowolski, 2003)} \]

- $\Rightarrow$ domain dependency of solution and approximation $\propto a$
Standard regularity result (if $\partial \Omega \in C^2(\Omega)$ or polygonal in 2D):

$$\|u\|_2 + \|p\|_1 < \frac{c}{L} (\|f\| + \|g\|_1)$$

Estimate also important for Aubin/Nitsche-Duality!

Desired improvement:

$$|u|_2 + |p|_1 < c (\|f\| + \|g\|_1)$$

and $c$ independent of $L$.

- **Problem:** Proof still requires an estimate of $|u|_1$
- $\Rightarrow$ assumptions on $g$ needed ( $\Rightarrow$ locally balanced flow)
idea: Divide domain/problem in *harmless* parts

- $\Omega = \{\tilde{\Omega}_i\} \cap \Omega$,
- choose all $\Omega_i$ to be sphere-like $\Rightarrow$ harmless $L_i$.
- $\{\Phi_i\}$ partition of unity on $\{\tilde{\Omega}_i\}$.

**Definition (Local 2-Regularity)**

$\Omega$ is called local 2-regular iff the solution $(u_i, p_i) \in X \times Y$ of the local Stokes problems

$$-\Delta u_i + Dp_i = f_i, \quad \text{div} u_i = g_i \text{ in } \Omega_i, \quad u_i = 0 \text{ on } \partial \Omega_i$$

are in $H^2(\Omega_i)^n \times H^1(\Omega_i)$ and satisfy:

$$\|u_i\|_2 + \|p_i\|_1 \leq c_i(\|f_i\| + \|g_i\|_1)$$
Preparations (2)

Definition (locally balanced flow)

Stokes Equations have **locally balanced flow** iff there exists a partition \( \{\bar{\Omega}_i\} \) of \( \Omega \) with LBB constants \( L_i \) independent of the aspect ratio \( a \) of \( \Omega \) and iff \( g \) satisfies

\[
\int_{\Omega_i} g \, dx = 0 \quad \forall \Omega_i
\]

i.e. sources and drains lie close to each other

- \( c_{Du} \) is defined by:
  \[
  |u|_1 < c_{Du} (\|f\| + \|g\|)
  \]

- regularity constant of **worst** possible configuration in \( \Omega \)

\[
c_R = \sup_{g,f} (c_{Du})
\]
Proof idea:
- divide Ω into subdomains Ω_i with harmless L_i
- use partition of unity to do the proof locally on each Ω_i

Estimates of local Stokes problems:
\[
\|f_i\| \leq \frac{c}{L_i} \left( \|f\|_\Omega_i + \|u\|_{1;\Omega_i} \right),
\]
\[
\|g_i\| \leq c \|u\|_{1;\Omega_i} + c \|g\|_{1;\Omega_i},
\]
\[
\|u\phi_i\|_{2;\Omega_i}^2 + \|(p - p_i)\phi_i\|_{1}^2 \leq c \left( \|f\|_{\Omega_i}^2 + \|u\|_{1;\Omega_i}^2 + \|g\|_{1;\Omega_i}^2 \right).
\]

Summation over i this results in
\[
\|u\|_{2}^2 + |p|_{1}^2 \leq c \left( \|f\|_1^2 + \|g\|_{1}^2 + \|u\|_{1}^2 \right).
\]
Estimate of $Du$

\[ |u|_1^2 = (f, u) + (g, p), \quad \int_{\Omega_i} g = 0 \quad \forall i \]

Above precondition on $g$ allows us to insert $p_i := \int_{\Omega_i} p \, dx$

\[ (g, p) = \sum_i \int_{\Omega_i} (p - p_i) g \, dx \leq \sum_i \|p - p_i\|_{\Omega_i} \|g\|_{\Omega_i} \]

Since $p - p_i \in L^2_0(\Omega_i)$ we can apply the LBB condition on $p - p_i$:

\[ (g, p) \leq \sum_i \frac{1}{L_i} \|g\|_{\Omega_i} \sup_{v \in H_0^{1,2}(\Omega_i)} \frac{(\text{div} \, v, p - p_i)_{\Omega_i}}{|v|_{1,\Omega_i}} \]

\[ \leq c \left( \max_i 1/L_i \right) \|g\| (|u|_1 + \|f\|) \]
What happens without locally balanced flow?

- Problem extremely violates condition of locally balanced flow
- But partial improvement due to:

\[
\begin{align*}
  u(x, y) &\propto y^2 \quad \Rightarrow \quad |u|_1 \propto \sqrt{a} \quad \Rightarrow \quad c_{Du} \propto \frac{1}{\sqrt{L}} \\
  \|u\|_2 + |p|_1 &\leq \frac{c}{\sqrt{L}} (\|f\| + \|g\|_1)
\end{align*}
\]
FEM Setup

Solution \((u_h, p_h) \in X_h \times Y_h \cap Y\) of conforming FEM

\[
(Du_h, Dv_h) - (\text{div} \ v_h, p_h) = (f, v_h) \quad \forall v_h \in X_h
\]
\[
(\text{div} \ u_h, q_h) = (g, q_h) \quad \forall q_h \in Y_h
\]

satisfies the error relations

\[
(D(u - u_h), Dv_h) = (\text{div} \ v_h, p - p_h) \quad \forall v_h \in X_h
\]
\[
(\text{div}(u - u_h), q_h) = 0 \quad \forall q_h \in Y_h.
\]

For simplicity we again restrict to
- 2-regular problems
- first order FE
Approximation Operators

\[
\text{div}(v - \Pi_h v, q_h) = 0 \quad \forall q_h \in Y_h, \\
\|v - \Pi_h v\| \leq c h^k |v|_k \quad \forall k = 0..m, \\
|v - \Pi_h v|_l \leq c h^k |v|_{k+l} \quad \forall k = 1..m, \: l = 0, 1; \\
\|q - S_h q\| \leq c h^k |q|_k \quad \forall k = 0..m, \\
|q - S_h q|_1 \leq c |q|_1,
\]

- We restrict to \( m = 2 \), e.g. operators exist for Mini-Element.
- First condition involves

\[
(\text{div} \: \Pi_h u - u_h, q_h) = 0 \quad \forall q_h \in Y_h.
\]

- Existence of discrete LBB constant \( L_h \) is guaranteed
Theorem

1. Assuming above approximation operators $\Pi_h, S_h$, the discrete solution satisfies

$$|u - u_h|_1 \leq ch(|u|_2 + |p|_1).$$

2. If in addition $\Omega$ is locally 2-regular, then we obtain

$$\|u - u_h\| \leq ch|u - u_h|_1 + ch^2|p|_k \leq ch^2(|u|_2 + |p|_1).$$

If $\Omega$ is locally 2-regular and if the Stokes problem has locally balanced flow, we obtain the LBB-free error bounds

$$|u - u_h|_1 \leq ch(||f|| + ||g||_1)$$

$$\|u - u_h\| \leq ch^2(||f|| + ||g||_1)$$
Proof Sketch

\[ |u - u_h|^2_1 = (D(u - u_h), D(u - \Pi_h u)) + (\text{div}(\Pi_h u - u_h), p - S_h p) \]
\[ \leq |u - \Pi_h u|_1 |u - u_h|_1 + (|u - \Pi_h u|_1 + |u - u_h|_1) \|p - S_h p\| \]
\[ \leq \frac{1}{2} |u - u_h|^2_1 + ch^2 (|u|^2_2 + |p|^2_2). \]

To approach the \( L^2 \)-estimate, we define the dual problem with solution \((\omega, \phi) \in X \times Y\) according to Aubin/Nitsche:

\[(Dv, D\omega) - (\text{div} \, v, \phi) = (u - u_h, v) \quad \forall v \in X, \]
\[(\text{div} \, \omega, q) = 0 \quad \forall q \in Y.\]

Dual problem is locally 2-regular, due to \( g \equiv 0 \):

\[ \|\omega\|_2 + |\phi|_1 \leq c \|u - u_h\|. \]
Proof Sketch (2)

Again we insert approximation operators at appropriate positions:

\[
\|u - u_h\|^2 = (D(u - u_h), D(w - \Pi_h\omega)) + (\text{div} \, \Pi_h\omega, p - p_h) - (\text{div}(u - u_h), \phi - S_h\phi).
\]

Using

\[
(\text{div} \, \omega_h, p - p_h) = (\text{div}(\Pi_h\omega - \omega), p - S_hp) \leq |\Pi_h\omega - \omega|_1 \|p - S_hp\|,
\]

we obtain

\[
\|u - u_h\|^2 \leq |u - u_h|_1 ch (|\omega|_2 + |\phi|_1) + ch^2 |\omega|_2 |p|_1
\]

\[
\leq c \|u - u_h\| (h |u - u_h|_1 + h^2 |p|_1)
\]

\[
\|u - u_h\| \leq ch^2 (\|u\|_2 + |p|_1).
\]
Pressure Error in Dual Norms

- immediate calculation shows
  \[ \| D(p - p_h) \|_{-1} \leq c h (|u|_2 + |p|_1) \]

- no surprise, due to LBB-condition
  \[ \| p - p_h \| \leq \frac{1}{L} \| D(p - p_h) \|_{-1} \]

- \( \| p - p_h \|_{-1} \) requires dual problem for arbitrary \( \tilde{g} \)
  \[ (D\omega, D\phi) + (\text{div} \phi, q) = 0 \quad \forall \phi \in X, \]
  \[ (\text{div} \omega, \psi) = (\tilde{g}, \psi) \quad \forall \psi \in Y, \]

- only partial improvement from \( \frac{1}{L} \) to \( c_R \)
  \[ \| p - p_h \|_{-1} \leq c c_R h^2 (|u|_2 + |p|_1) \]
technical computation yields inverse inequality
\[ \forall q_h \in Y_h \subset H^1(\Omega): \]
\[ \| Dq_h \| \leq c h^{-1} \| Dq_h \|_{-1,h;\Omega} := \sup_{v_h \in X_h} \frac{(v_h, Dq_h)}{|v_h|_1} \]

as a corollary one obtains
\[ |p - p_h|_1 \leq |p - S_h p|_1 + |S_h p - p_h|_1 \]
\[ \leq c |p|_1 + c h^{-1} \| D(S_h p - p) + D(p - p_h) \|_{-1,h;\Omega} \]
\[ \leq c |p|_1 + c h^{-1} \sup_{v_h \in X_h} \frac{(Dv_h, D(u - u_h))}{|v_h|_1} \]
\[ \leq c(|p|_1 + |u|_2). \]
Dual problem has no locally balanced flow:

\[(D\omega, D\phi) + (\text{div} \phi, q) = 0 \quad \forall \phi \in X,\]

\[(\text{div} \omega, \psi) = (p - p_h, \psi) \quad \forall \psi \in Y.\]

standard computation using approximation operators yields:

\[\|p - p_h\|_2^2 \leq (\text{div}(\omega - \Pi_h \omega), p - p_h) + (D(u - u_h), D\Pi_h \omega)\]

\[\leq |\omega - \Pi_h \omega|_1 \cdot \|D(p - p_h)\|_1 + \]

\[+ (D(u - u_h), D(\Pi_h \omega - \omega)) + (\text{div}(u - u_h), q - S_h q)\]

\[\leq c \left( |\omega - \Pi_h \omega|_1 + \|D(q - S_h q)\|_1 \right) \left( |u - u_h|_1 + \|D(p - p_h)\|_1 \right)\]

\[\leq c c_R^2 h^2 \left( |p|_1^2 + |u|_2^2 \right)\]
**L²-Error Bound and LBB-friendly Methods**

\[ \| p - p_h \| \leq c c_R h (|p|_1 + |u|_2) \]

- partial improvement depending on the domain
- Idea for further improvement:
  - use discrete solution \((\omega_h, q_h)\) instead of \(\Pi_h q, S_h q\)
  \[
  \| p - p_h \|^2 \leq c \left( |\omega - \omega_h|_1 + \|D(q - q_h)\|_{-1} \right) \times \\
  \times \left( |u - u_h|_1 + \|D(p - p_h)\|_{-1} \right)
  \]
  - use LBB-friendly FE approximation

- \(L^2\)-pressure error independent of \(L\) if

\[
|u - u_h|_1 + \|D(p - p_h)\|_{-1} \leq c h (\|f\| + \|g\|_1)
\]

is independent of \(L\) for arbitrary right-hand side \((f, g)\)
very bad case models free channel flow with zero-boundary

Subtract free flow solution $\tilde{u}_{0,h}(y)$ from original problem to roughly achieve locally balanced flow!

What to subtract in complex configurations?

Idea: Let the FEM do the job!
Advice: Choose FEM such that the FE space contains a free flow solution which corresponds to a Stokes problem with right-hand side \((\tilde{f}, \tilde{g})\) and \(\tilde{g}\) satisfying

\[
\int_{\Omega} (g - \tilde{g}) dx = 0
\]

- Example: FEM does subtraction automatically if free flow solution is an element of FE space (e.g. Mini-Element)

- Take care of stabilization terms:

\[
(div \, u_h, q_h) - \sum_{\Lambda} c_{\Lambda} \mu(\Lambda) \int_{\Lambda} (f - Dp_h) Dq_h dx = (g, q_h)
\]
**Worst Case**

\[
|u - u_h|_1 + \|D(p - p_h)\|_\infty \leq c c_{Du} (\|f\| + |g|_1) \\
\|u - u_h\| \leq c c_{Du} (\|f\| + |g|_1) \\
\|p - p_h\| \leq c c_{Rc_{Du}} (\|f\| + |g|_1) \\
\Rightarrow \ c_{Rc_{Du}} \text{ instead of } \frac{1}{L^2}
\]

**Locally-balanced flow & LBB-friendly Methods**

1. locally balanced flow \( \Rightarrow \ c_{Du} \text{ independent of } L \\
2. appropriate method \( \Rightarrow \ c_{R} (\text{almost}) \text{ independent of } L \\

😊 All error bounds (almost) independent of LBB constant \( L \)!
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