Discretization of Elliptic Differential Equations Using Sparse Grids and Prewavelets

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Example of a high dimensional problem:

$$- \Delta u + cu = f \quad \text{on} \quad \Omega = ]0,1[^6, \quad u|_{\partial \Omega} = 0$$

$$c(\vec{x}) = \prod_{i=1}^{6} \sin(\pi x_i)$$

Application: Schrödinger equation

Standard FE methods are inefficient:

Computational amount is $O(N^d)$ to calculate FE solution $u_h$, where $N = h^{-1}$
Sparse Grids

- Number of grid points:
  \[ |D_n| = O\left(N \left(\log N\right)^{d-1}\right) \]
  \[ N = 2^n \]
  \[ d : \text{dimension} \]

- Interpolation error:
  \[ \| u - I_{\text{sparse}}(u) \|_{H^1} = O\left(N^{-1} \left(\log N\right)^{d-1}\right) \]
  \[ \| u - I_{\text{sparse}}(u) \|_{L^2} = O\left(N^{-2} \left(\log N\right)^{d-1}\right) \]
  \[ \| u - I_{\text{sparse}}(u) \|_{L^\infty} = O\left(N^{-2} \left(\log N\right)^{d-1}\right) \]

[Zenger, GAMM-Seminar, 1990]
Basis Functions on a Sparse Grids

sparse grid: \( D_n \)

support of bilinear FE functions:

space: \( V_{D_n} \subset H_0^1(\Omega) \)
**FE Galerkin discretization:** [Zenger, Bungartz, Balder]

Find \( u_n \in V_{D_n} \)

\[
a(u_n, v) = \int f v \, dx \quad \forall v \in V_{D_n}
\]

where \( a(u, v) = \int \nabla u \nabla v \, dx \)

\( V_{D_n} \subset H_0^1(\Omega) \)
Overlapping Basis Functions on a Sparse Grids

In case of variable coefficients $a(v_p, v_q)$ are $O(N^2)$ data.

How to compute: $a(v_p, v_q)$?
History

- Sparse grids for **constant coefficients**:  
  - Zenger, Bungartz, Balder, ..., SISC 1996, ..., SISC 2015

- **variable coefficients in 2D**: semi-orthogonality concept:  
  - [complete convergence theory](#)  
  - fast iterative solver, Q-cycle  
  - only 2D!

- Other approaches for variable coefficients:  
  - simulation results in 3D  
  - no proof of convergence!  
  - non-symmetric discretization, difficulties with multigrid  
  - high order interpolations are needed for Achatz’s approach
Sparse grids and Galerkin approach applied to hierarchical basis functions could not be applied to problems like:

\[- \Delta u + cu = f \quad \text{on} \quad \Omega = ]0,1[^6, \quad u \big|_{\partial \Omega} = 0\]

\[c(\tilde{x}) = \prod_{i=1}^{6} \sin(\pi x_i)\]
Sparse Grids with Prewavelets

\[-\Delta u + cu = f \quad \text{on} \quad \Omega = ]0,1[^6, \quad u \big|_{\partial \Omega} = 0\]

\[c(\vec{x}) = \prod_{i=1}^{6} \sin(\pi x_i)\]

Simulation results using sparse grids, prewavelets and semi-orthogonality:

<table>
<thead>
<tr>
<th>Depth ( n )</th>
<th>( | u - u_{D_n} |_{\infty} = e_n )</th>
<th>( e_{n-1} / e_n )</th>
<th>Number unknowns</th>
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<tbody>
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Orthogonality Property of Prewavelets

**Theorem**

Hat functions are orthogonal to coarse grid functions with respect to

$$ (u, v) \rightarrow \int_{0}^{1} \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} \, dx $$

Prewavelets are orthogonal to coarse grid functions with respect to

$$ (u, v) \rightarrow \int_{0}^{1} uv \, dx $$
**Theorem**

Let \( a(u, v) = \int \nabla u \nabla v + cuv \, dx \)

\( H^1_0 \)-elliptic where \( c \) constant.

Then,

\[ a(v_p, v_q) = 0 \]

where \( v_p, v_q \) are overlapping and

\( v_p, v_q \) are

hat functions for \( c = 0, d = 2 \)

or

prewavelets for every \( d \geq 2 \).
**Discretization (for variable coefficients)**

Let \( a(u, v) = \int \nabla u^T A \nabla v + cuv \, dx \)

\( H^1_0 \) - elliptic where \( A, c \) variable coefficients.

Then, define

\[
a_n^{so}(v_p, v_q) = \begin{cases} 
0 & \text{if } v_p, v_q \text{ are overlapping} \\
 a(v_p, v_q) & \text{else}
\end{cases}
\]

Find \( u_n \in V_{D_n} \) such that

\[
a_n^{so}(u_n, v) = \int fv \, dx \quad \forall v \in V_{D_n}
\]
Convergence for Helmholtz Problem, Variable Coefficient

Convergence of Discretization with Semi-Orthogonality

Let \( a(u, v) = \int \nabla u^T \nabla v + cuv \, d\bar{x} \) and \( c(\bar{x}) \geq 0 \) smooth coefficient.

Let \( u_n \in V_{D_n} \) such that

\[
a_n^{so}(u_n, v) = \int f_v \, d\bar{x} \quad \forall v \in V_{D_n}
\]

Let \( u \in H^1_0 \) such that

\[
a(u, v) = \int f_v \, d\bar{x} \quad \forall v \in H^1_0
\]

\( h = 2^{-n} \). Then, there is a constant \( K > 0 \) such that

\[
\|u - u_n\|_{H^1} \leq Kh(\log(h^{-1}))^{d-1}
\]

\( O(h...) \)
Proof of Convergence

One has to show:

\[ |a_n^{so}(w_n, v_n) - a(w_n, v_n)| \leq C \left\| w_n \right\|_{H^1} \left\| v_n \right\|_{H^1} 2^{-n_d} n^{d-2} \]

where \( n_d = \begin{cases} n & \text{if } d \leq 4 \\ \frac{n}{2} & \text{if } d \geq 5 \end{cases} \)

for every \( w_n, v_n \in V_{D_n} \)

To this end, one mainly has to show:

\[ \left| \int c(\bar{x}) v_p v_q d \bar{x} \right| \leq C \left\| v_p \right\|_{H^1} \left\| v_q \right\|_{H^1} 2^{-n_d} n^{d-2} \]

for every overlapping basis functions \( v_p, v_q \)
Proof of Convergence

Use constant interpolation of $c$ on overlapping domain:

$$\left| \int c(\bar{x}) v_p v_q d\bar{x} \right| = \left| \int (c - I(c))(\bar{x}) v_p v_q d\bar{x} \right| \leq C \left\| v_p \right\|_{H^1} \left\| v_q \right\|_{H^1} 2^{-n_d} n^{d-2}$$

for every overlapping basis functions $v_p, v_q$

where $I(c)$ interpolates $c$ only in those directions,

where $p, q$ have different levels.

$v_p \in V_{D_n}$

no points inside

$v_q \in V_{D_n}$
**Level of Subgrids of a Sparse Grids**

- **$n=2$**
  - $t_2 = 2$
  - $t_2 = 1$
  - $t_2 = 0$

- **$t = (t_1, t_2)$**
  - $(1,1)$
  - $(2,0)$

**Definition**

Let $t, t' \in \mathbb{N}_0^d$ be the levels of basis functions $v_p$ and $v_q$. $v_p, v_q$ are **overlapping** if:

$$|t| \leq n, \quad |t'| \leq n, \quad |\max(t, t')| > n$$
Lemma:

Let $t, t' \in \mathbb{N}_0^d$ be the level of the basis functions $\nu_p$ and $\nu_q$.

$|t| \leq n, \quad |t'| \leq n, \quad m := |\max(t, t')| > n$

Then the following inequality holds:

$$\sum_{s \in \Theta_{t,t'}} \max(t_s, t'_s) + \max(t) + \max(t') \geq (m - n) + n_d$$

where $n_d := \begin{cases} \frac{2}{d - 2} & \text{if } d \leq 4 \\ n & \text{if } d \geq 5 \end{cases}$

Counter example in $d=5$:

$$t = \left(1, 0, \frac{n}{3} - 1, \frac{n}{3}, \frac{n}{3}\right)$$

$$t' = \left(0, 1, \frac{n}{3} - 1, \frac{n}{3}, \frac{n}{3}\right)$$
Convergence for Variable Diffusion Coefficient

Convergence of Discretization with Semi-Orthogonality

Let \( a(u, v) = \int \nabla u^T A \nabla v d\tilde{x} \) an \( H^1 \)-elliptic bilinear form.

Let \( u_n \in V_{D_n} \) such that
\[
a^s_n(u_n, v) = \int f v d\tilde{x} \quad \forall v \in V_{D_n}
\]

Let \( u \in H^1_0 \) such that
\[
a(u, v) = \int f v d\tilde{x} \quad \forall v \in H^1_0
\]

\( h = 2^{-n} \). Then, there is a constant \( K > 0 \) such that
\[
\| u - u_n \|_{H^1} \leq K h (\log(h^{-1}))^{d-1} \quad \text{if } d \leq 3
\]

\( O(h...) \) if \( d \leq 3 \)

Plus additional assumptions!

\( d \geq 4 \) can be analyzed too.
Poisson Problem on Curvilinear Bounded Domain

\[-\Delta u = f \quad \text{on} \quad \Omega \subset ]0,1[^3, \quad u \mid_{\partial \Omega} = g\]

\[u(\vec{x}) = \prod_{i=1}^{3} \sin(\pi x_i)\]
Poisson Problem on Curvilinear Bounded Domain

\[-\Delta u = f \quad \text{on} \quad \Omega \subset ]0,1[^3, \quad u \Big|_{\partial \Omega} = g\]

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</table>
Iterative Solver and Implementation

- **iterative solver**
  - diagonal preconditioned cg-algorithm
  - small condition number

- **implementation of stiffness matrix vector multiplication:**
  - implementation of $2^d$ cases:
    - in each direction: restriction or prolongation
    - do first prolongation and then restrictions
  - restriction and prolongation based on standard 1-dimensional formulas
    - stencil: ( 0, $\frac{1}{2}$, 1, $\frac{1}{2}$, 0 )
  - 1-dimensional formula for prewavelet coefficients (1/10, -3/5, 1, -3/5, 1/10)
  - standard $3^d$ – stencil (in 3D: 27-stencil and in 6D: 729-stencil)
new discretization of elliptic differential equations

- elliptic PDE’s with variable coefficients in d-dimensional space
- sparse grids
- prewavelets
- semi-orthogonality property (*this makes everything easier!*)

properties:

- efficient algorithm for matrix vector multiplication
- simulation results for a 3D and 6D problem
- convergence proof in case of variable coefficients.
Thank you for your attention!