Energy corrected schemes in flow problems for optimal control

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H. Egger (TU Darmstadt)

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Outline

- Motivation
- Optimal convergence of FE approximations in the presence of corner singularities
  - Scalar elliptic
  - Stokes equation
  - Optimal control
- Conclusions
Model problem. Consider the Poisson problem

\[-\Delta u = f \quad \text{in } \Omega\]
\[u = g \quad \text{on } \partial \Omega\]

on a polygonal domain with reentrant corner.

Weak formulation (g=0). Find \( u \in H^1_0(\Omega) \) such that

\[
\int_{\Omega} \nabla u \cdot \nabla v \, dx =: a(u, v) = (f, v) := \int_{\Omega} f v \, dx \quad \forall v \in H^1_0(\Omega)
\]

equivalent to minimization problem

\[
\frac{1}{2} a(u, u) - (f, u) = \min! \quad \text{on } H^1_0(\Omega)
\]

Lax-Milgram. There exists a unique weak solution \( u \in H^1_0(\Omega) \).
Pollution Effect for Elliptic PDE

Corner singularities: \( u \not \in H^2(\Omega) \) and pollution effect for standard FE

- L-shape domain: \( \theta = \frac{3}{2} \pi \)

<table>
<thead>
<tr>
<th>( l )</th>
<th>weighted ( L^2 )-norm</th>
<th>rate</th>
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<tr>
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- Slit domain: \( \theta = 2\pi \)

<table>
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<tr>
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Standard FE solutions do not recover best approximation order of \( O(h^2) \)
How to avoid the pollution effect

Standard remedies: • Graded meshes and adaptivity
• Enrichment by singular functions
• Weighted least squares formulation

Alternative approach: Energy correction [Zenger+Gietl 78, Rüde+Zenger 86]
Find \( u_l(\gamma) \in V_l \) such that \( a_\gamma(u_l(\gamma), v_l) = f(v_l), v_l \in V_l \) with

\[
a_\gamma(v_l, w_l) := a(v_l, w_l) - \gamma a_\omega(v_l, w_l)
\]

\( V_l \) is lowest order conforming finite element space associated with the mesh \( \mathcal{T}_l \)

Question: How to define \( \omega_l \)?
\( \omega_l \) local patch with fixed number of elements

\[
\bar{\omega}_l := \bigcup \{ T \in \mathcal{T}_l; \ x_c \in \partial T \}
\]

\( x_c \) is the re-entrant corner
How to avoid the pollution effect

**Standard remedies:**
- Graded meshes and adaptivity
- Enrichment by singular functions
- Weighted least squares formulation

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Find \( u_l(\gamma) \in V_l \) such that \( a_\gamma(u_l(\gamma), v_l) = f(v_l), \ v_l \in V_l \) with

\[
a_\gamma(v_l, w_l) := a(v_l, w_l) - \gamma a_{\omega_l}(v_l, w_l)
\]

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**Question:** How to define \( \omega_l \)?

\( \omega_l \) local patch with **fixed** number of elements

\[
\bar{\omega}_l := \bigcup \{ T \in T_l; \ x_c \in \partial T \}
\]

\( x_c \) is the **re-entrant** corner
Existence and Uniqueness

Interpretation of the modification for $\gamma < 1$: Local softening of the stiffness

**Theorem:** There exists a unique $\gamma_\infty(\theta)$ depending on the local coarse mesh such that

$$
\|u - u_l(\gamma_\infty(\theta))\|_{0;B} = \mathcal{O}(h_l^2), \quad B := \{x \in \Omega; \text{dist}(x_c, x) \geq d\}
$$

if the interior angle $\theta$ at $x_c$ satisfies $\pi < \theta \leq \frac{3}{2}\pi$ or the coarse mesh is symmetric.

**Observation:** Necessary condition for $\mathcal{O}(h_l^2)$ convergence in the weighted $L^2$-norm

$$
(*) \quad a(s_i, s_j) - a_\gamma((s_i)l(\gamma), (s_j)l(\gamma)) = \mathcal{O}(h_l^2), \quad i, j = 1, 2
$$

with the singular functions $s_i = \eta(r)r^{i\pi/\theta} \sin\left(\frac{i\pi}{\theta} \phi\right), \eta(r)$ smooth cut-off function.

**Theorem:** The condition $(*)$ is also **sufficient** for optimal order convergence.

If it holds for $i = j = 1$ and $\theta$ satisfies $\pi < \theta < \frac{3}{2}\pi$ or the coarse mesh is symmetric then it also holds for all $i, j = 1, 2$. 
Convergence rates

Energy correction yields **optimal** convergence order in weighted $L^2$-norm
Energy correction yields *optimal* convergence order in weighted $L^2$-norm.
Optimal convergence for the stress intensity factor

Kondratiev expansion: \( u = \sum_{0 < n < \frac{\theta}{\pi}} k_n s_n + U, \ k_n \in \mathbb{R}, \ U \in H^2(\Omega) \)

Stress-intensity factors \( k_n \) are given by

\[
k_n = \frac{1}{n\pi} \left( \int_{\Omega} f s_{-n} + u \Delta s_{-n} \, dx \right).
\]

**Theorem:** stress-intensity factors converge with \( |k_n - k_n^h| = \mathcal{O}(h^2) \)

L-shape domain: analytical \( k_1 = 1 \), error given in percent

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<th>rate</th>
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How can we find the correction factor $\gamma_\infty$?

C. Waluga, B. Wohlmuth, UR (2014): Nested Newton strategies for energy-corrected finite elements at corner singularities, SISC

Define the **energy defect** function by:

$$g_l(\gamma) := a(u, u) - a(u_l(\gamma), u_l(\gamma)) + \gamma a_w(u_l(\gamma), u_l(\gamma))$$

**Theorem:** On each level there exists a unique $\gamma^*_l < 1$ such that $g_l(\gamma^*_l) = 0$.

**Remark:** The energy defect function is strictly **monotone**.

We have $g_l(\gamma_\infty) = O(h_l^2)$.

On each level, we get the relation

$$g_{l-1}(\gamma) = 2^{2\pi/\theta} g_l(\gamma) + O(h_l^2)$$

**Theorem:** The root $\gamma^*_l$ of $g_l(\cdot)$ defines a Cauchy sequence. Moreover

$$|\gamma^*_l - \gamma_\infty| \leq C h_l^{2(1 - \pi/\theta)}$$

and **optimal convergence** rates are obtained

$$\|u - u_l(\gamma^*_l)\|_{0;B} = O(h_l^2)$$
Convergence study for energy corrected method

**Slit domain:** weighted error norm

$$\left\| r^{1/2}(s - R_h s) \right\|_{L^2(\Omega)}$$

<table>
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<th>level</th>
<th>$\gamma = 0$</th>
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<th>rate</th>
<th>$\gamma = \gamma_i^*$</th>
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Using extrapolation to approximate $\gamma_\infty$

L-shape domain ($\theta = \frac{3}{2}\pi$)

Domain with a slit ($\theta = 2\pi$)

Optimal parameter: $\gamma_n = \gamma_\infty + c_1 h_n^{2-2\pi/\theta} + \ldots$
Adaptive refinement leads to graded meshes, same convergence order, but worse cost constant and each DOF more expensive.
Generalizations of the energy correction


- Many re-entrant corners
- Higher order FE: several correction parameters are necessary
- Elliptic systems (elasticity)
  - structure of singularities more complex
  - several correction parameters necessary
- Jumpy coefficients (4 corner problem, checkerboard domain): works well
- Eigenvalues
- 3D is difficult: structure of singularities more complex
The increase of convergence rates on level 5 can be explained by the fact that we can achieve significantly improved results. Here, both the uncorrected solution and also the meshes. The results are shown in Table 5.4. Additionally, the solution on level 3 is examined on a series of uniformly refined meshes. The algorithms are based on Newton's method and can be embedded into finite elements without energy correction. Moreover, the given fit provides a good approximation of the energy correction, since the standard finite elements do not need to be further investigated. Moreover, we see that for the positions of the crack-tips, respectively. Fig. 5.5 (left) illustrates the geometry and flux errors for the domain with several cracks.

### Table 5.4: Error in Flux through Boundary

<table>
<thead>
<tr>
<th>l</th>
<th>standard finite elements</th>
<th>energy-corrected finite elements</th>
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<tbody>
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</tr>
<tr>
<td>5</td>
<td>5.373e-4</td>
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</table>

- $\Delta u = 0$, $u = \frac{1}{4} \cos(\pi x_2) + 1$, $u = \frac{1}{4} \cos(\pi x_2)$
- $\nabla u \cdot n = 0$
- $\bar{\Omega} = [-2, 2] \times [-1, 1]$
- Error in Flux through boundary:

$$\| h^{1/2} \nabla(u - u_h) \cdot n \|_{L^2(\Gamma_2)}.$$
Domain with idealized „cracks“ \( \overline{\Omega} = [-2, 2] \times [-1, 1] \)

\[-\Delta u = 0, \]
\[u = \frac{1}{4} \cos(\pi x_2) + 1 \]
\[u = \frac{1}{4} \cos(\pi x_2) \]
\[\nabla u \cdot \mathbf{n} = 0 \]

in \( \Omega \),
on \( \Gamma_1 := \{-2\} \times (-1, 1) \)
on \( \Gamma_2 := \{2\} \times (-1, 1) \),
on \( \partial \Omega \setminus (\Gamma_1 \cup \Gamma_2) \).

### Table 5.4

<table>
<thead>
<tr>
<th>l</th>
<th>weighted</th>
<th>rate</th>
<th>flux</th>
<th>rate</th>
<th>weighted</th>
<th>rate</th>
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<td>6.325e-7</td>
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</tr>
</tbody>
</table>

Error in Flux through boundary:
\[ \| h^{1/2} \nabla (u - u_h) \cdot \mathbf{n} \|_{L^2(\Gamma_2)}. \]
$$-\Delta u = 0,$$

$$u = \frac{1}{4} \cos(\pi x_2) + 1$$

$$u = \frac{1}{4} \cos(\pi x_2)$$

$$\nabla u \cdot n = 0$$

in $\Omega$, on $\Gamma_1 := \{-2\} \times (-1, 1)$

on $\Gamma_2 := \{2\} \times (-1, 1)$, on $\partial \Omega \backslash (\Gamma_1 \cup \Gamma_2)$.

Error in Flux through boundary:

$$\| h^{1/2} \nabla (u - u_h) \cdot n \|_{L^2(\Gamma_2)}.$$

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</table>

Energy correction for controlled PDE  -  Ulrich Rüde
Energy corrected FEM for Stokes equation


Let $\Omega \subset \mathbb{R}^2$ be bounded, polygonal, $f \in L^2_{-\alpha}(\Omega)^2$

$$-\Delta u + \nabla p = f \quad \text{in } \Omega,$$

$$\text{div } u = 0 \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on } \partial \Omega.$$

Characterization of solutions: let $\omega \in (\pi, 2\pi)$ and $1 - \beta < \text{Re}(\lambda) < 1 + \alpha$, then

$$u = \sum_{i=1}^{N} \sum_{j=1}^{I_i} \sum_{k=0}^{\kappa_{i,j} - 1} c_{i,j,k} s_{i,j,k}^u + U,$$

$$p = \sum_{i=1}^{N} \sum_{j=1}^{I_i} \sum_{k=0}^{\kappa_{i,j} - 1} c_{i,j,k} s_{i,j,k}^p + P$$

with constants $c_{i,j,k}$ and

$$s_{i,j,k}^u = \eta(r) r^{\lambda_i} \sum_{l=0}^{k} \frac{1}{l!} (\ln r)^l \left( \varphi_{j,k-l}^{(i)}(\theta) \right),$$

$$s_{i,j,k}^p = \eta(r) r^{\lambda_i - 1} \sum_{l=0}^{k} \frac{1}{l!} (\ln r)^l \xi_{j,k-l}^{(i)}(\theta),$$

with $(U, P) \in H^2_{-\alpha}(\Omega)^2 \times H^1_{-\alpha}(\Omega)$

[Dauge 1989], [Ortl, Sändig 1995]
Structure of Singularities

$\lambda_i$ have to satisfy the condition

$$\lambda_i^2 \sin^2(\omega) - \sin^2(\lambda_i \omega) = 0, \quad \lambda_i \neq 0, \quad \lambda_i \in \mathbb{C}.$$
Structure of Singularities

Singular functions for the L-shaped domain $\omega = 3/2\pi$

\[
    u = \sum_{i=1}^{2} \eta(r) r^{\lambda_i} (\varphi^{(i)}(\theta), \psi^{(i)}(\theta))^\top + U, \quad p = \sum_{i=1}^{2} \eta(r) r^{\lambda_i - 1} \xi^{(i)}(\theta) + P
\]

Figure: Sing. functions $(s_{i,r}^u, s_{i,\theta}^u, s_i^p)$ for $i = 1, 2$
Energy correction for Stokes equation


**Variational formulation** for **stable FE pair** (e.g. Mini, P2–P1, ...)

Find \((u_h, p_h) \in V_h \times Q_h\):

\[
a_h(u_h, v_h) + b(v_h, p_h) = \langle f, v_h \rangle_{\Omega} \quad \forall v_h \in V_h,
\]

\[
b(u_h, q_h) = 0 \quad \forall q_h \in Q_h,
\]

with \(a_h(u_h, v_h) := a(u_h, v_h) - \gamma_1 a_{\omega_1}(u_h, v_h) - \gamma_2 a_{\omega_2}(u_h, v_h)\).

**Variational formulation** for **stabilized FE** (e.g. P1–P1 with PSPG stab.)

Find \((u_h, p_h) \in V_h \times Q_h\):

\[
a_h(u_h, v_h) + b(v_h, p_h) = \langle f, v_h \rangle_{\Omega} \quad \forall v_h \in V_h,
\]

\[
b(u_h, q_h) - c_h(p_h, q_h) = 0 \quad \forall q_h \in Q_h,
\]

with \(c_h(p_h, q_h) := c(p_h, q_h) - (1 - \gamma_1)^{-1} c_{\omega_1}(p_h, q_h) - (1 - \gamma_2)^{-1} c_{\omega_2}(p_h, q_h),\)

→ motivated from the equivalence of Mini element and PSPG.
Numerical Example: L-shaped domain

Error in velocity with standard method and with correction

without correction

with correction
Local modification - desired global effect

difference between energy corrected and standard FE solution

error in the energy corrected (left) and in the standard (right) FE solution
Optimal boundary control of the Stokes equations

\( \Omega \subset \mathbb{R}^2 \), with **Control**, **Dirichlet** (inflow/no-slip) and **Neumann** (outflow) boundaries, 
\( \Gamma = \Gamma_C \cup \Gamma_D \cup \Gamma_N \) and given desired state \( \bar{u} \in L^2(\Omega_{\text{obs}})^2 \) with \( \Omega_{\text{obs}} \subset \Omega \).

**Minimize** the cost functional

\[
J(u, z) := \frac{1}{2} \| u - \bar{u} \|^2_{L^2(\Omega_{\text{obs}})} + \frac{1}{2} \| z \|^2_{H^{1/2}_0(\Gamma_C)},
\]

subject to the constraint

\[-\nu \Delta u + \nabla p = 0 \quad \text{in } \Omega,\]

\[\text{div } u = 0 \quad \text{in } \Omega,\]

\[u = z \quad \text{on } \Gamma_C,\]

\[u = 0 \quad \text{on } \Gamma_D,\]

\[\nu (\nabla u) n - p n = 0 \quad \text{on } \Gamma_N.\]

**Remark**

Similar for \( f \in H^{-1}(\Omega)^2 \) and \( g \in H^{1/2}(\Gamma_D)^2 \).
Optimal boundary control of the Stokes equations

Optimality system (after formal elimination of the control $z$) [John 2014]

\[-\nu \Delta u + \nabla p = 0 \quad \text{in } \Omega, \quad -\nu \Delta w - \nabla r = \chi_{\text{obs}}(u - \bar{u}) \quad \text{in } \Omega,\]

\[\text{div } u = 0 \quad \text{in } \Omega, \quad \text{div } w = 0 \quad \text{in } \Omega,\]

\[\varrho(\nu(\nabla u)n - p n) = t(w, r) \quad \text{on } \Gamma_C, \quad w = 0 \quad \text{on } \Gamma_C,\]

\[u = 0 \quad \text{on } \Gamma_D, \quad w = 0 \quad \text{on } \Gamma_D,\]

\[\nu(\nabla u)n - p n = 0 \quad \text{on } \Gamma_N, \quad \nu(\nabla w)n + r n = 0 \quad \text{on } \Gamma_N,\]

with $t(w, r) = \nu(\nabla w)n + r n$.

Remark
The control can be obtained in a post processing step by $z = u|_{\Gamma_C}$. 
Energy corrected FE discretization

For the discretization we consider Taylor–Hood elements, $V_h \times Q_h \subset H^1_0(\Omega, \Gamma_D)^2 \times L^2(\Omega)$ and $	ilde{V}_h := \{v_h \in V_h : v_h = 0 \text{ on } \Gamma_C \} \subset V_h$.

Variational formulation
Find $(u_h, p_{z,h}, w_h, r_h) \in V_h \times Q_h \times \tilde{V}_h \times Q_h$:

$$
\begin{align*}
\tilde{a}_h(u_h, v_h) + \tilde{b}(v_h, p_{z,h}) + a_h(w_h, v_h) + b(v_h, r_h) &= \langle \chi_{obs}\bar{u}, v_h \rangle_\Omega & \forall v_h \in V_h, \\
\tilde{b}(u_h, q_h) + b(w_h, q_h) &= 0 & \forall q_h \in Q_h, \\
a_h(u_h, \tilde{v}_h) + b(\tilde{v}_h, p_{z,h}) &= 0 & \forall \tilde{v}_h \in \tilde{V}_h, \\
b(u_h, \tilde{q}_h) &= 0 & \forall \tilde{q}_h \in Q_h
\end{align*}
$$

with $a_h(u_h, v_h) := a(u_h, v_h) - \gamma_1 a_{\omega_1}(u_h, v_h) - \gamma_2 a_{\omega_2}(u_h, v_h)$,
$\tilde{a}_h(u_h, v_h) := \langle \chi_{obs}u_h, v_h \rangle_\Omega + \varrho a_h(u_h, v_h)$ and $\tilde{b}(u_h, q_h) = \varrho b(u_h, q_h)$.

Observation
The asymptotic corrections parameters $\gamma = \gamma^*$ form the Stokes equations with Dirichlet bc’s work.
Optimal boundary control of the Stokes equations

Example 2 Domain with multiple re-entrant corners

Fig.: Domain with different boundary parts.

Computations with the following data:

- desired state, changing with the flow direction, i.e.
  \[ \bar{u} = \begin{cases} (0, -5/2)^T & \text{for } x_1 \in [1, 3/2] \\ (0, 5/2)^T & \text{else.} \end{cases} \]

- cost coefficient \( q = 10^{-4} \)

- viscosity constant \( \nu = 1 \)
Optimal boundary control of the Stokes equations

Example 2 Domain with multiple re-entrant corners
Optimal boundary control of the Stokes equations

Fig.: State $u_h^m$ for different cost coefficients $\varrho \in \{10^{-2}, 10^{-3}, 10^{-4}\}$
Optimal boundary control of the Stokes equations

Fig.: State $u^m_h$ for different cost coefficients $\varrho \in \{10^{-5}, 10^{-6}\}$

<table>
<thead>
<tr>
<th>$\frac{1}{2} | u^m_h - \bar{u} |<em>{L^2(\Omega</em>{obs})}^2$</th>
<th>$10^{-1}$</th>
<th>$10^{-2}$</th>
<th>$10^{-3}$</th>
<th>$10^{-4}$</th>
<th>$10^{-5}$</th>
<th>$10^{-6}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{2} | u^m_h - \bar{u} |<em>{L^2(\Omega</em>{obs})}^2$</td>
<td>0.35582</td>
<td>0.23986</td>
<td>0.07697</td>
<td>0.05947</td>
<td>0.05808</td>
<td>0.05179</td>
</tr>
</tbody>
</table>

$\mathcal{J}(u^m_h, z^m_h)$

| $\mathcal{J}(u^m_h, z^m_h)$ | $0.36539$ | $0.29812$ | $0.13399$ | $0.06874$ | $0.05966$ | $0.05492$ |

Tab.: Values of the cost functional for different cost coefficients $\varrho$. 

Energy correction for controlled PDE - Ulrich Rüde
Optimal boundary control of the Stokes equations

Example 3 Domain with multiple re-entrant corners

Fig.: Domain with different boundary parts.

Computations with the following data:

- desired state, parabolic profile, i.e.
  \[ \bar{u} = 4x_2(1 - x_2), \]

- cost coefficient \( \varphi = 10^{-4} \)

- viscosity constant \( \nu = 1 \)

- \( \Omega_{\text{obs}} = (2.4, 3.5) \times (0, 1) \)
Optimal boundary control of the Stokes equations

Example 3 Domain with multiple re-entrant corners
Conclusions

- Standard FE on uniform meshes do not provide optimal order rates.

- Energy-corrected FE require only a local modification of stiffness matrix.

- The accuracy order of many quantities of interest, e.g.,
  - weighted $L^2$-norms (errors)
  - eigenvalues, stress intensity factors
  cannot be better than the accuracy in the energy.

- The number of required correction parameters depends on the desired order and the singular functions of the solution.

- Improved performance in case of boundary controlled flow problems shown.
Thank you for your attention!

see also B.Wohlmuth’s invited lecture Wed 11-12
and UR’s talk on particulate flow MS-Tu-E-27-2 16:30-17:00
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